PRODUCTS OF POINCARÉ DOMAINS[1]

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ABSTRACT. A domain $\Omega \subseteq \mathbb{R}^N$ of finite N-dimensional Lebesgue measure is a p-Poincaré domain ($1 \leq p \leq \infty$) if there exists a positive constant K such that the p-Poincaré inequality: $\|u\|_{L^p(\Omega)} \leq K\| \bigtriangledown u\|_{L^p(\Omega)}$, is valid for all Sobolev functions $u \in W^{1,p}(\Omega)$ which integrate to zero. Define $K_p(\Omega)$ to be the smallest such K if Ω is a p-Poincaré domain, and to be infinity otherwise. We obtain comparability relations between $K_p(\Omega_1 \times \Omega_2)$ and the pair: $K_p(\Omega_1), K_p(\Omega_2)$. In particular, our results show that p-Poincaré domains are closed under cartesian products (for all p), and that in case p equals 2, we have $K_2(\Omega_1 \times \Omega_2) = \max\{K_2(\Omega_1),\ K_2(\Omega_2)\}$.

§1 INTRODUCTION

Throughout this paper a "domain" $\Omega \subseteq \mathbb{R}^N$ will be an open connected set having finite N-dimensional Lebesgue measure which we denote by $|\Omega|$. All "functions" are assumed to be real-valued and measurable. For an integrable function h on Ω we let h_Ω denote its average value, *i.e.*,

$$h_\Omega = \frac{1}{|\Omega|}\int_\Omega h(x)\ dx.$$

For convenience of establishing some notation, we include the definition of the Sobolev spaces. More information on the Sobolev spaces and their functions can be found in: **[Zie-89],** Chapter 7 of **[Gil&Tr-83], [Maz-85],** and **[Ad-75]**.

Definition. For $u, v \in L^1_{loc}(\Omega)$ (meaning that u and v are functions on Ω which are integrable on compact subsets) and $1 \leq i \leq N$, we say that v is the i^{th} <u>weak partial derivative</u> of u on Ω (written as: $v = D^i u$) provided that the <u>integration by parts formula</u>

$$\int_\Omega v(x)\varphi(x)dx = -\int_\Omega u(x)\frac{\partial}{\partial x_i}\varphi(x)dx$$

is valid for each $\varphi \in \mathcal{C}^1_o(\Omega)$, the class of continuously differentiable functions having compact support in Ω. If a weak derivative exists, it follows from the Lebesgue differentiation theorem that it is uniquely determined almost everywhere. If $D^1u, \cdots, D^Nu$ all exist, we write $\bigtriangledown u$ for the <u>weak gradient</u> $(D^1u, \cdots, D^Nu)$ of u.

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[1]The results of this paper come from the author's doctoral dissertation **[Sta-90]** done at the University of Michigan under the direction of Peter L. Duren and Allen L. Shields.

For a number p, $1 \le p \le \infty$, we define the (first order) Sobolev space

$$W^{1,p}(\Omega) \equiv \{u \in L^p(\Omega) : \bigtriangledown u \in L^p(\Omega)\} \ .$$

We are now prepared to define Poincaré domains, the principal objects under investigation.

Definition. The best p-Poincaré constant $(1 \le p \le \infty)$ of Ω is given by

$$K_p(\Omega) = \sup\left\{\frac{\|u - u_\Omega\|_{L^p(\Omega)}}{\| \bigtriangledown u\|_{L^p(\Omega)}} : u \in W^{1,p}(\Omega), u \neq \text{constant}\right\}. \tag{1}$$

The domain Ω is said to be a p-Poincaré domain if $K_p(\Omega)$ is finite. Thus p-Poincaré domains are those domains for which there exists a positive number K such that the p-Poincaré inequality:

$$\|u - u_\Omega\|_{L^p(\Omega)} \le K\| \bigtriangledown u\|_{L^p(\Omega)} \tag{2}$$

is valid for all $u \in W^{1,p}(\Omega)$.

The Poincaré inequalities are prototypes for general Sobolev inequalities which are important in the theories of partial differential equations and mathematical physics. The problem of estimating the best p-Poincaré constants and in particular of classifying (geometrically) the p-Poincaré domains has recently received a great deal of attention, see, *e.g.*, [**Sm&St-90**], [**Mar-88**], [**Sm&St-87**], [**Ev&Ha-87**], and [**Jer-86**].

The purpose of this paper is to analyze the relationship between the best p-Poincaré constants of the cartesian product of a pair of domains and those of the individual factor domains. The formula

$$K_p(\Omega_1 \times \Omega_2) \ge \max\{K_p(\Omega_1),\ K_p(\Omega_2)\} \tag{3}$$

results from the fact that for each function $u(x) \in W^{1,p}(\Omega_1)$, the function $\widetilde{u}(x,y) \equiv u(x)$ is in $W^{1,p}(\Omega_1 \times \Omega_2)$ and the quotients appearing in (1) which correspond to these two functions are the same. Formula (3) implies that the cartesian product of a non-p-Poincaré domain with any domain whatsoever is again a non-p-Poincaré domain. A natural question thus arises: is the cartesian product of a pair of p-Poincaré domains again a p-Poincaré domain? Our main result is the following theorem which, in particular, yields an affirmative answer to this question.

Theorem 1.1. *Suppose that N_1 and N_2 are positive integers. For domains $\Omega_1 \subseteq \mathbb{R}^{N_i}$ we have*

$$K_p(\Omega_1 \times \Omega_2) \le \begin{cases} 2^{(p-1)/p}\sqrt[p]{K_p(\Omega_1)^p + K_p(\Omega_2)^p} & [p < \infty] \\ K_\infty(\Omega_1) + K_\infty(\Omega_2) & [p = \infty], \end{cases} \tag{4}$$

and

$$K_2(\Omega_1 \times \Omega_2) = \max\{K_2(\Omega_1),\ K_2(\Omega_2)\} \ . \tag{5}$$

Our proof of (4) is based on real variable methods and will be given in Section 3. It is possible to give a similar proof of (5). Instead of this, however, we give in Section 4 a proof of (5) which is based on a useful connection with the spectral theory of unbounded operators. In Section 2 we prove some lemmas which will be needed in Section 3. The paper concludes with Section 5 in which we discuss the problem of sharpening the estimate (4) and present some open questions.

§2 PRELIMINARY LEMMAS

We begin with a rather technical separability result which we use only in this section.

Lemma 2.1. *Let $\Omega \subseteq \mathbb{R}^N$ be a domain and let j be an integer, $1 \le j \le N$. There exists a countable subset $D_j \subseteq \mathcal{C}_0^1(\Omega)$ such that to any $\varphi \in \mathcal{C}_0^1(\Omega)$, there corresponds a sequence $\langle \varphi_n \rangle \subseteq D_j$ with*

$$\int \varphi f = \lim \int \varphi_n f \quad \text{and} \quad \int \left(\frac{\partial}{\partial x_j} \varphi \right) f = \lim \int \left(\frac{\partial}{\partial x_j} \varphi_n \right) f$$

for each $f \in L^1(\Omega)$

Proof. We begin by exhausting Ω by an increasing sequence of compact subdomains $\langle \Omega_k \rangle_{k=1}^{\infty}$ with the further property that $\overline{\Omega}_k \subseteq \Omega_{k+1}$ for each k. Let $\psi_k : \Omega \longrightarrow [0,1]$ be a function in $\mathcal{C}_0^1(\Omega)$ which is supported in Ω_{k+1} and which is identically equal to 1 on Ω_k (see for example **[Hör-63]**, page 4).

The support of each function $\varphi \in \mathcal{C}_0^1(\Omega)$ will eventually be contained in some Ω_k. It follows from the Stone-Weierstrass theorem that the polynomials in $\mathbb{Q}[x_1, x_2, \cdots, x_n]$, i.e., those with rational coefficients, are uniformly dense in $\mathcal{C}(\overline{\Omega_{k+1}})$. Thus for each $\varepsilon > 0$, there exists a polynomial $\psi \in \mathbb{Q}[x_1, x_2, \cdots, x_n]$ such that

$$\sup_{\Omega_{k+1}} \left| \frac{\partial}{\partial x_j} \varphi - \psi \right| \quad \le \quad \frac{\varepsilon}{1 + \operatorname{diam}(\overline{\Omega_{K+1}})} . \tag{6}$$

The same inequality persists if we replace ψ by $\psi_k \psi$. Now consider the function

$$\widetilde{\psi} : \mathbb{R}^N \longrightarrow \mathbb{R}^1$$

defined by

$$\widetilde{\psi}(x_1, \cdots, x_n) = \int_{-\infty}^{x_j} (\psi_k \psi)(x_1, \cdots, x_{j-1}, t, x_{j+1}, \cdots, x_n) dt .$$

Since the support of $\psi_k \psi$ lies in Ω_{k+1}, we infer from (6) that

$$\sup_{\Omega_{k+1}} \left| \varphi - \widetilde{\psi} \right| \le \varepsilon \quad \text{and} \quad \sup_{\Omega_{k+1}} \left| \frac{\partial}{\partial x_j} \varphi - \frac{\partial}{\partial x_j} \widetilde{\psi} \right| \le \varepsilon .$$

The function $\widehat{\psi} = \psi_k \widetilde{\psi}$ satisfies the above inequalities with $C(k)\varepsilon$ in place of ε and has the further property that it is in $\mathcal{C}_0^1(\Omega)$. Consequently, for a fixed $f \in L^1(\Omega)$, we have that

$$\left| \int (\varphi - \widehat{\psi}) f \right| \le \sup_{\Omega_{k+1}} \left| \varphi - \widehat{\psi} \right| \quad \|f\|_{L^1(\Omega_{k+1})} \le (\text{CONSTANT}) \varepsilon ,$$

where the CONSTANT depends only on k (and f) whereas ε can be chosen arbitrarily small. We can obtain a similar upper bound for

$$\left| \int \frac{\partial}{\partial x_j} (\varphi - \widehat{\psi}) f \right| .$$

The lemma is now proved by letting D_j be the countable set of all functions $\widehat{\psi}$ obtained as above. □

The next two lemmas give ways to construct new Sobolev functions from old ones. The first lemma asserts that almost every slice of a Sobolev function is a Sobolev function; the latter shows that averages of Sobolev functions are Sobolev functions.

Lemma 2.2. *Suppose that $1 \le p \le \infty$, that $\Omega_i \subseteq \mathbb{R}^{N_i} (i = 1, 2)$ are domains, and that $u \in W^{1,p}(\Omega_1 \times \Omega_2)$. Then for almost every $x \in \Omega_1$, the function*

$$u_x : \Omega_2 \longrightarrow \mathbb{R}^1 :: y \longmapsto u(x, y)$$

lies in $W^{1,p}(\Omega_2)$ and moreover, for $1 \le j \le N_2$ we have

$$D^j(u_x) = (D^j_{\Omega_2} u)_x \ . \tag{7}$$

Proof. Fubini's theorem implies that for almost every $x \in \Omega_1$, the functions u_x and $(D^j_{\Omega_2} u)_x$ are in $L^p(\Omega_2)$. Thus, we need only check that (7) is valid for almost every $x \in \Omega_1$.

We fix an index j, $1 \le j \le N_2$, and let $\psi \in \mathcal{C}^1_0(\Omega_1)$ and $\varphi \in \mathcal{C}^1_0(\Omega_2)$. Then

$$\Phi(x, y) \equiv \psi(x)\varphi(y) \quad \in \quad \mathcal{C}^1_0(\Omega_1 \times \Omega_2) \ ,$$

and using Fubini's theorem we obtain

$$\begin{aligned} &\int_{\Omega_1} \psi(x) \left[\int_{\Omega_2} \frac{\partial}{\partial y_j} \varphi(y) u_x(y) dy \right] dx \\ &= \int_\Omega (D^j_{\Omega_2} \Phi) u \\ &= -\int_\Omega \Phi(D^j_{\Omega_2} u) \\ &= \int_{\Omega_1} \psi(x) \left[-\int_{\Omega_2} \varphi(y) (D^j_{\Omega_2} u)_x(y) dy \right] dx \ . \end{aligned}$$

Thus the bracketed functions of x appearing in the first and last of the above integrals must be equal in the sense of distributions on Ω_1 as ψ is allowed to range over all $\mathcal{C}^1_0(\Omega_1)$ functions. However, Fubini's theorem implies that the two bracketed functions are both locally integrable on Ω_1. Consequently, to each $\varphi \in \mathcal{C}^1_0(\Omega_2)$ there corresponds a set $G_\varphi \subseteq \Omega_1$ of full measure on which the bracketed functions agree, *i.e.*, $x \in G_\varphi$ implies

$$\int_{\Omega_2} \frac{\partial}{\partial y_j} \varphi(y) u_x(y) dy = -\int_{\Omega_2} \varphi(y) (D^j_{\Omega_2} u)(y) dy \ . \tag{8}$$

Next, we let D_j be a countable subset of $\mathcal{C}^1_0(\Omega_2)$ as in Lemma 2.1. Set $G = \cap_{\psi \in D_j} G_\psi$. It follows that (8) holds uniformly for every $x \in G$ and every $\varphi \in \mathcal{C}^1_0(\Omega_2)$. Put differently, this says that for each $x \in G$, the identity (7) is valid. $\square$

Lemma 2.3. *Suppose that $\Omega_i \subseteq \mathbb{R}^{N_1} (i = 1, 2)$ are domains and that $u \in W^{1,p}(\Omega)$, where $\Omega = \Omega_1 \times \Omega_2$ and $1 \le p \le \infty$. Then the function*

$$U_1 : \Omega_1 \longrightarrow \mathbb{R}^1$$

defined by

$$x \longmapsto (u_x)_{\Omega_2} \equiv \frac{1}{|\Omega_2|} \int_{\Omega_2} u(x, y) dy$$

lies in $W^{1,p}(\Omega)$, and for each $j (i \le j \le N_1)$ we have

$$D^j U_1(x) = \frac{1}{|\Omega_2|} \int_{\Omega_2} D^j_{\Omega_1} u(x, y) dy. \tag{9}$$

The proof of Lemma 2.3 is similar to that of Lemma 2.2; we omit it. The final result which we need here is an inequality which can be viewed as a vector-valued analogue of the familiar fact (for scalar valued functions) that the L^p-norm of a function dominates the absolute value of the integral of the function, when $p \ge 1$.

Lemma 2.4. *Let $\langle X, \mathbb{S}, \mu\rangle$ be a probability space, $f_j \in L^1(\mu)$ $(1 \le j \le N)$ and let $1 \le p < \infty$. Then*

$$\left[\sum_{j=1}^N \left|\int_X f_j(x)d\mu(x)\right|^2\right]^{p/2} \quad \le \quad \int_X \left[\sum_{j=1}^N |f_j(x)|^2\right]^{p/2} \quad d\mu(x) \ .$$

Lemma 2.4 is a special case of the vector valued Jensen's inequality which can be stated as follows (see page 55 of **[Nev-85]** for a proof). Suppose $D \subseteq \mathbb{R}^n$ is a convex domain and $\Phi : D \longrightarrow \mathbb{R}$ is a convex function. Then for any probability space $\langle X, \mathbb{S}, \mu\rangle$ and vector valued function $\overrightarrow{F} : X \longrightarrow D$ we have

$$\Phi\left(\int_X \overrightarrow{F}(x)d\mu(x)\right) \quad \le \quad \int_X \Phi\left(\overrightarrow{F}(x)\right) d\mu(x) \ .$$

Lemma 2.4 is then just this result with $\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by $\Phi(x) = |x|^p = (\Sigma_{i=1}^n x_i^2)^{p/2}$.

§3 PROOF OF THE GENERAL INEQUALITY

Using the notation and results of Section 2, we proceed to give a proof of the estimate (4) of Theorem 1.1.

Case 1. $p < \infty$

Let $u \in W^{1,p}(\Omega)$. Fubini's theorem gives us that

$$\int_\Omega |u - u_\Omega|^p = \int_{\Omega_1}\int_{\Omega_2} |u(x,y) - u_\Omega|^p dy \ dx \ .$$

We invoke the elementary inequality: $(a+b)^p \le 2^{p-1}(a^p + b^p)$, which is valid for $1 \le \ p < \ \infty$ and a,$b \ge 0$, to conclude that the above integral is

$$\le \int_{\Omega_1}\int_{\Omega_2} 2^{p-1}\left[|u_x(y) - (u_x)_{\Omega_2}|^p + |(u_x)_{\Omega_2} - u_\Omega|^p\right] dydx \ .$$

Lemma 2.2 permits us to estimate this integral as

$$\le 2^{p-1}\int_{\Omega_1}\left[K_p(\Omega_2)^p\int_{\Omega_2} |\bigtriangledown u_x|^p dy + |\Omega_2||U_1(x) - u_\Omega|^p\right] dx \ .$$

We infer from Lemma 2.2 that $|\bigtriangledown u_x(y)| \le |\bigtriangledown u(x,y)|$. Using this inequality in conjunction with Lemma 2.3, we can estimate the preceding integral as

$$\le 2^{p-1}\left[K_p(\Omega_2)^p\int_\Omega |\bigtriangledown u|^p + |\Omega_2|K_p(\Omega_1)^p\int_{\Omega_1} |\bigtriangledown U_1|^p dx\right].$$

Finally, in light of the formula for $\bigtriangledown U_1$ resulting from (9), we can invoke Lemma 2.4 to estimate the last integral as

$$\le 2^{p-1}\left[K_p(\Omega_1)^p + K_p(\Omega_2)^p\right]\int_\Omega |\bigtriangledown u|^p \ .$$

Case 2. $p = \infty$

A few modifications in the above proof of Case 1 yield a proof that works here as well. $\square$

§4 SPECTRAL THEORY APPROACH TO THE CASE $p = 2$

Here we prove the identity (5) of Theorem 1.1. For clarity we shall first extract from the proof several preliminary results. Throughout this section we will assume familiarity with the standard notations and basic principles of the theory of unbounded operators, as is done, for example, in **[Kat-76]**, or in **[Re&Sim-I-80]** and **[Re&Sim-IV-80]**.

For a domain $\Omega \subseteq \mathbb{R}^N$, we let q denote the gradient (quadratic) form on Ω with domain $D(q) = W^{1,2}(\Omega)$, *i.e.*,

$$q(\varphi, \psi) = (\nabla\varphi, \nabla\psi) \text{ for } \varphi, \psi \in W^{1,2}(\Omega)\,,$$

where $(\cdot,\ \cdot)$ is the usual inner product on $\mathbb{R}^N$. The corresponding Friedrichs extension operator is then called the Neumann Laplacian (written as $-\Delta_{N,\Omega}$ or $-\Delta$ for short). The Neumann Laplacian is intimately connected with the 2-Poincaré inequality on Ω as is shown by the following result.

Proposition 4.1. *A domain $\Omega \subseteq \mathbb{R}^N$ is a 2-Poincaré domain if and only if there is a gap in the spectrum of the Neumann Laplacian $-\Delta_{N,\Omega}$ between zero and the positive elements of the spectrum. In any case, the infimum of the positive part of the spectrum in $\sigma(-\Delta_{N,\Omega})$ coincides with the square of reciprocal of the best 2-Poincaré constant of $\Omega, K_2(\Omega)$.*

I am very grateful to the referee for pointing out to me Section 10 of **[Den&Li-53]** as the original source for this beautiful and useful connection. Some comments are in order. The Laplace operator used in this paper coincides with the Neumann Laplacian. This is easily seen in one direction by the uniqueness property of Friedrichs extensions and in the other direction by a simple integration by parts. Also, the result is only actually stated and proved under the supplementary assumption that the inclusion operator $W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$ is compact.

This assumption was only used in the application of the minimax principle to locate the bottom of the positive spectra of $-\Delta$. The minimax principle is, however, valid for all self-adjoint operators (see page 75 *ff* of **[Re&Sim-IV-80]** and also page 338 therein for an interesting history of this result.) The minimax principle had been known in the compact case since 1910 but was only extended to the general self-adjoint case by Kato at about the time of that **[Den&Li-53]** was written. In the present situation, the eigen vector $u(x) \equiv 1$ corresponds to the "ground state" of $-\Delta$. The minimax principle then tells us that the infimum of the positive part of $\sigma(-\triangle)$ is given by

$$\sup_{\varphi} \inf_{\substack{\psi \in D(-\Delta) \\ \|\psi\|=1 \\ \psi \perp \varphi}} (-\Delta\psi, \psi) = \inf_{\substack{\psi \in D(-\Delta) \\ \|\psi\|=1 \\ \int_\Omega \psi(x)dx = 0}} (-\Delta\psi, \psi)\ .$$

Theorem 4.2. *Let Ω_1 and Ω_2 be arbitrary domains in $\mathbb{R}^{N_1}$ and $\mathbb{R}^{N_2}$ respectively, and let Ω denote the product domain $\Omega_1 \times \Omega_2 \subseteq \mathbb{R}^{N_1+N_2}$. We denote the corresponding Neumann Laplacians by $-\Delta_1, -\Delta_2$, and $-\Delta$, respectively. If we identify $L^2(\Omega)$ "naturally" with $L^2(\Omega_1) \otimes L^2(\Omega_2)$, we then have the following identity:*

$$-\Delta = \overline{((-\Delta_1) \otimes I + I \otimes (-\Delta_2))|_{D(-\Delta_1)\otimes D(-\Delta_2)}}\ .$$

Proof. Let us use R to denote the operator $(-\Delta_1)\otimes I + I\otimes(-\Delta_2)$ on the domain $D(-\Delta_1)\otimes D(-\Delta_2)$. It follows from Exercise VIII.47 of **[Re&Sim-I-80]** that R is essentially self-adjoint, in other words, $\overline{R}$ is self-adjoint. Since $-\Delta$ is also self-adjoint, and self-adjoint operators are maximal symmetric operators (*i.e.*, if A is self-adjoint B is symmetric and $A \subseteq B$ then $A = B$. Proof: $A = A^* \supseteq B^* \supseteq B.\square$) it suffices to show that these two operators agree on a typical tensor $\varphi\otimes\psi \in D(-\Delta_1)\otimes D(-\Delta_2)$.

By uniqueness of Friedrichs extensions, it is enough to show that $\varphi\otimes\psi (= \varphi(x)\cdot\psi(y))$ lies in $D(q)(W^{1,2}(\Omega))$ and for each $\Phi \in W^{1,2}(\Omega)$, we have $q(\varphi\otimes\psi,\Phi) = (R(\varphi\otimes\psi),\Phi)_{L^2(\Omega)}$.

Fubini's theorem clearly yields that $\varphi\otimes\psi \in D(q)$. We adopt the following notation: $\Phi_x(y) = \Phi(x,y) = \Phi_y(x)$, where Φ is any function on Ω. The apparent ambiguity inherent in this notation is transcended by the fact that x and y will not be substituted by any other variables. Now let $\Phi \in W^{1,2}(\Omega)$. Recall that from Lemma 2.2, for a.e. $y \in \Omega_2$, $\Phi_y \in W^{1,2}(\Omega_1)$; and symmetrically, $\Phi_x \in W^{1,2}(\Omega_2)$ for a.e. $x \in \Omega_1$. By invoking the properties of Friedrichs extensions, we obtain

$$
\begin{aligned}
&(R(\varphi\otimes\psi), \quad \Phi)_{L^2(\Omega)}\\
&\quad= (-\Delta_1\varphi(x)\psi(y) - \varphi(x)\Delta_2\psi(y) \quad , \quad \Phi(x,y))_{L^2(\Omega)}\\
&\quad= \int_{\Omega_2}\psi(y)(-\Delta_1\varphi,\Phi_y)_{L^2(\Omega_1)}dy + \int_{\Omega_1}\varphi(x)(-\Delta_2\psi,\Phi_x)_{L^2(\Omega_2)}dx\\
&\quad= \int_{\Omega_2}\psi(y)(\bigtriangledown_x\varphi,\bigtriangledown_x\Phi_y)_{L^2(\Omega_1)}dy + \int_{\Omega_1}\varphi(x)(\bigtriangledown_y\psi,\bigtriangledown_y\Phi_x)_{L^2(\Omega_2)}dx\\
&\quad= \int_{\Omega_1}\int_{\Omega_2}\left[\psi(y)\bigtriangledown_x\varphi(x)\cdot\bigtriangledown_x\Phi(x,y) + \varphi(x)\bigtriangledown_y\psi(y)\cdot\bigtriangledown_y\Phi(x,y)\right]dydx\\
&\quad= \iint_{\Omega_1\times\Omega_2}\bigtriangledown(\varphi\otimes\psi)\cdot\bigtriangledown\Phi\\
&\quad= q(\varphi\otimes\psi,\Phi)\ .
\end{aligned}
$$

The proof is thus complete. $\square$

The following result, which is a special case of the corollary on page 301 of **[Re&Sim-I-80]**, will be the final ingredient we will need for the proof of (5).

Proposition 4.3. *If A_i is a self-adjoint operator with domain $D(A_i) \subseteq \mathcal{H}_i (i = 1,2)$, then*

$$
\sigma\left[\overline{(A_1\otimes I + I\otimes A_2)|_{D(A_1)\otimes D(A_2)}}\right] = \overline{\sigma(A_1)+\sigma(A_2)}\ .
$$

Proof of (5). In this proof we shall be using the notations for the three Neumann Laplacians which were given in Theorem 4.2. Proposition 4.1 tells us that

$$
\frac{1}{\lambda}\ K_2\ (\Omega_1\times\Omega_2) = \sup\left\{\frac{1}{\lambda}\mid\lambda\in\sigma(-\Delta)\setminus\{0\}\right\}^{1/2}\ .
$$

By Theorem 4.2 and Proposition III.15, we can rewrite this as

$$
\begin{aligned}
&= \sup\left\{\frac{1}{\lambda_1+\lambda_2}\mid\lambda_i\in\sigma(-\Delta_i)\text{ not both zero}\right\}^{1/2}\\
&= \max\{K_2(\Omega_1)\ ,\ K_2(\Omega_2)\}\quad\square
\end{aligned}
$$

§5 CONCLUDING REMARKS AND QUESTIONS

Theorem 1.1 suggests the following questions.

Question 5.1. For $1 \leq p \leq \infty$, $p \neq 2$, what is the smallest constant $\Lambda(p)$ for which

$$K_p(\Omega_1 \times \Omega_2) \leq \Lambda(p) \max \{K_p(\Omega_1),\ K_p(\Omega_2)\} \ ?$$

We believe that $\Lambda(p)$ is realized in the simple case in which $\Omega_1 =]-1,\ 1[= \Omega_2$. The author has calculated all of the best p-Poincaré constants of $]-1,\ 1[$ (see **[Sta-90]**), they are given as follows:

$$K_p(\ ]-1,1[\ ) = \begin{cases} 1 & \text{if } p = 1, \infty \\ \frac{p \sin(\pi/p)}{\pi (p-1)^{1/p}} & \text{if } 1 < p < \infty \ . \end{cases}$$

The formula (5) makes it seem plausible that perhaps $\Lambda(p)$ equals 1 for all p. In case $p = \infty$, we can consider the function $f :\ ]-1,\ 1[^2 \longrightarrow \mathbb{R}^1$ defined by

$$(x, y) \longmapsto y + x \ ,$$

to find that $K_\infty \left( ]-1,\ 1[^2 \right) \geq 2^{1/2} > 1 = K_\infty \left( ]-1, 1[\right)$ so that (5) cannot be true for K_∞ in place of K_2 (and in fact, by continuity of L^p norms, it cannot be true for any sufficiently large p). At the other extreme, *i.e.*, when $p = 1$, the problem is quite a bit more recalcitrant:

Question 5.2. : Is $K_1 \left( ]-1,\ 1[^2 \right)$ larger than $1 \,(= K_1 \left( ]-1, 1[\right))$?

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