SMOOTH APPROXIMATION OF SOBOLEV FUNCTIONS ON PLANAR DOMAINS

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Abstract. We examine two related problems concerning a planar domain Ω. The first is whether Sobolev functions on Ω can be approximated by global C^∞ functions, and the second is whether approximation can be done by functions in $C^\infty(\Omega)$ which, together with all derivatives, are bounded on Ω. We find necessary and sufficient conditions for certain types of domains, such as starshaped domains, and we construct several examples which show that the general problem is quite difficult, even in the simply connected case.

1. INTRODUCTION

Let Ω be a bounded domain (i.e., a bounded open connected set) in $\mathbb{R}^n$. For a number p, $1 \leq p < \infty$, and a nonnegative integer k, we let $W^{k,p}(\Omega)$ denote the Sobolev space of those (real-valued) L^p-integrable functions u on Ω whose distributional partial derivatives $D^\alpha u$, $|\alpha| \leq k$, also lie in $L^p(\Omega)$. The Sobolev space $W^{k,p}(\Omega)$ becomes a Banach space when endowed with the norm

$$\|u\|_{W^{k,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(\Omega)}.$$

By a *Sobolev function*, we mean any function which belongs to some Sobolev space. The Sobolev functions are the "functions" of the modern theories of partial differential equations and consequently have been extensively studied for their own intrinsic properties. For good general references on the Sobolev spaces and their functions, we cite: [13], [1], [10], and Chapter 7 of [5]. We employ the familiar notation $C^\infty(\Omega)$ for the space of infinitely differentiable functions on Ω and $C^\infty(\overline{\Omega})$ for the space of restrictions of functions in C^∞ $(= C^\infty(\mathbb{R}^n))$ to Ω. We also let $C_b^\infty(\Omega)$ denote those $C^\infty(\Omega)$-functions u for which $D^\alpha u$ is bounded on Ω for all α. Of course, $C^\infty(\overline{\Omega})$ is contained in $C_b^\infty(\Omega)$ provided that Ω is a bounded domain.

A fundamental approximation theorem for Sobolev spaces is the well known result of Meyers and Serrin [11] which states that $C^\infty(\Omega)$ is dense in $W^{k,p}(\Omega)$ for all $k \geq 1$ and $1 \leq p < \infty$. In this paper, we study the questions of which domains satisfy the following closely related properties:

Property 1. $C^\infty(\overline{\Omega})$ is dense in $W^{k,p}(\Omega)$.

Property 2. $C_b^\infty(\Omega)$ is dense in $W^{k,p}(\Omega)$.

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Property 2 is a weaker requirement than Property 1. Property 1 has been shown to hold for domains satisfying the *segment condition* which says that, to every point $x \in \partial\Omega$, there corresponds a positive number η_x and nonzero vector y_x with the following property: if $|z - x| < \eta_x$ and $z \in \overline{\Omega}$ then Ω contains the open segment $\{z + ty_x : 0 < t < 1\}$. Indeed, if Ω satisfies the segment condition, then $C^\infty(\overline{\Omega})$ is dense in $W^{k,p}(\Omega)$ for all values of k and p (see, Theorem 3.18 on page 54 of [1]; also see [2]). Another class of domains for which Property 1 has an affirmative answer are the *uniform domains.*

Definition. A domain Ω is a uniform domain if and only if there is a constant M_Ω such that: given any two points $x, y \in \Omega$ there is a rectifiable curve γ in Ω with endpoints $\{x, y\}$ and such that:

(1.1) The length of gamma, $|\gamma|$, is less than $M_\Omega|x - y|$, and
(1.2) the distance to $\partial\Omega$ for any point $z \in \gamma$ is larger than M_Ω^{-1} times the shorter of the two arc lengths of $\gamma(x, z)$ and $\gamma(z, y)$.

Indeed, Jones introduced a class of subdomains of $\mathbb{R}^n$ which he called *ϵ–δ domains* and showed [6] that every Sobolev function on such a domain is the restriction of a global Sobolev function (of the same order and exponent). Hence, Meyers and Serrin approximation, applied to $\mathbb{R}^n$, implies that Property 1 holds on ϵ–δ domains. When the parameter δ is infinity, the corresponding class of domains coincides with the class of uniform domains. For a proof of this latter fact, see Theorem 2.10 in [12]; see also the proof of Theorem 7 in [4].

Our main results concern bounded *starshaped* domains in the plane, that is domains for which there is a point $x_0 \in \Omega$ such that for all $x \in \Omega$ the line segment $[x_0, x] \subset \Omega$, and *interior segment* domains.

Definition. A bounded domain Ω in $\mathbb{R}^n$ is called an interior segment domain, if to every point $x \in \partial\Omega$, there corresponds a positive number η_x and nonzero vector y_x with the following property: if $|z - x| < \eta_x$ and $z \in \Omega$ then Ω contains the segment $\{z + ty_x : 0 \le t \le 1\}$.

Definition. Let μ be a positive measure on $\mathbb{R}^n$. If $x \in \mathbb{R}^n$ and $E \subset \mathbb{R}^n$ satisfies $\mu(E \cap B(x, r)) > 0$ for every ball, centered at x with arbitrary radius $r > 0$, then we say that x is a μ-limit point of E.

It may at first appear that there is very little difference between the segment condition and the interior segment condition. However, the segment condition implies that $\partial\Omega$ is locally the graph of a continuous function, whereas it is possible that the n-dimensional Lebesgue measure of the boundary, $m_n(\partial\Omega)$, is positive for a domain satisfying the interior segment property. In fact, this possibility is a crucial aspect of our work. The following three theorems are the main results of this paper.

Theorem A. *Let Ω be a bounded domain in $\mathbb{R}^2$. If Ω is starshaped or if Ω satisfies the interior segment condition, then $C_b^\infty(\Omega)$ is dense in $W^{k,p}(\Omega)$ for all $k = 1, 2, \ldots$ and all $1 \le p < \infty$.*

Theorem B. *Let Ω be a bounded domain in $\mathbb{R}^2$ which is either starshaped or satisfies the interior segment condition. If every $z \in \partial\Omega$ is an m_2-limit point of Ω^c, then $C^\infty(\overline{\Omega})$ is dense in $W^{k,p}(\Omega)$ for all $k = 1, 2, \ldots$ and all $1 \le p < \infty$.*

Here Ω^c denotes the complement $\mathbb{R}^2 \setminus \Omega$. The next theorem shows that the converse to Theorem B holds in much greater generality. In particular, it shows that the density condition involving m_2-limit points of Ω^c, is necessary for $C^\infty(\overline{\Omega})$ to be dense in $W^{k,p}(\Omega)$, for finitely connected domains Ω.

Theorem C. *Let Ω be a domain in $\mathbb{R}^2$. Suppose that $z \in \partial\Omega$, that z is not an m_2-limit point of Ω^c and that z is a limit point of the set of nondegenerate components of $\partial\Omega$; then $C^\infty(\overline{\Omega})$ is not dense in $W^{k,p}(\Omega)$, for any $k \geq 1$ and any p, $1 \leq p < \infty$.*

Example 1. Let R denote the open rectangle $(0,2) \times (0,1)$ and let E be any proper closed subset of $[0,1]$. Put $\Omega = R \setminus ([0,1] \times E)$. It follows that Ω is an interior segment domain and hence $C_b^\infty(\Omega)$ is dense in $W^{k,p}(\Omega)$, by Theorem A. Moreover, by Theorem B and Theorem C, a necessary and sufficient condition that C^∞ is dense in $W^{k,p}(\Omega)$ is that every point of E is an m_1-limit point for E.

We construct an example showing that Theorem A and Theorem B both fail in dimension 3, and as a result we focus our attention on planar domains. Since uniform domains have Property 1 and since bounded simply connected planar uniform domains are the same as quasidisks, Jones asked whether every Jordan domain has Property 1, see [7]. Recently, Lewis [9] answered this question in the affirmative for the first order Sobolev spaces, i.e., for $k = 1$. On the other hand, there are remarkably simple examples for which $C^\infty(\overline{\Omega})$ is not dense $W^{k,p}(\Omega)$, but $\mathbb{R}^2 \setminus \overline{\Omega}$ is *not* connected; see [3] and [8]. In light of the above, it might seem reasonable that if Ω is a bounded simply connected planar domain, $\mathbb{R}^2 \setminus \overline{\Omega}$ is connected and each point of Ω^c is an m_2-limit point of Ω^c, then $C^\infty(\overline{\Omega})$ is dense in $W^{k,p}(\Omega)$. This turns out to be very far from the truth. We construct several examples in Section 7 which show that this is in fact false. Each of the examples are simply connected domains but the presence of what we call *two-sided boundary points* is a key feature. For a domain Ω, we say that a point $x \in \partial\Omega$ is a two-sided boundary point for Ω provided that there is a $\delta(x) > 0$ such that for each δ, $0 < \delta < \delta(x)$, the set $B(x,\delta) \cap \Omega$ has at least two components whose closures contain x. A better question is the following:

Question 1. If Ω is a simply connected planar domain without two-sided boundary points and for which all points in Ω^c are m_2-limit points of Ω^c, then does it follow that $C^\infty(\overline{\Omega})$ is dense in $W^{k,p}(\Omega)$?

In Sections 2 and 4 we prove special cases of Theorem A and Theorem B, with some preliminary work for the latter being in Section 3. The proofs Theorem A and Theorem B are found in Section 5. Section 6 contains Theorem C as well as another necessary condition for the approximation of Sobolev functions, and the examples mentioned above appear in Section 7.

2. Bounded Meyers and Serrin Approximation

The approximation result of Meyers and Serrin states that $C^\infty(\Omega)$ is dense in $W^{k,p}(\Omega)$, where Ω is any domain in $\mathbb{R}^n$. However, it is not always possible to approximate a Sobolev function with a $C^\infty(\Omega)$ function which has bounded derivatives on Ω. We construct a domain for which $C_b^\infty(\Omega)$ is not dense in $W^{k,p}(\Omega)$ in Section 7. Our main result in this section is a geometric condition which guarantees this type of approximation.

We first describe a *basic interior segment domain.* Let Ψ be a domain in $\mathbb{R}^{n-1}$, $I = (a,b)$ be a finite interval and $\eta > 0$. Denote by e_1 the first coordinate unit vector. We say that a domain Ω which is contained in the product $I \times \Psi$ is a *basic* *interior segment domain* provided $(b-\eta, b) \times \Psi \subset \Omega$ and for all other points $x \in \Omega$, $x + te_1$ is also in Ω, for $0 \leq t \leq \eta$. For example, the unit square in $\mathbb{R}^2$ minus the product $[0, 1/2] \times E$ is a basic interior segment domain for *any* closed set $E \subset [0, 1]$. See also Figure 1.

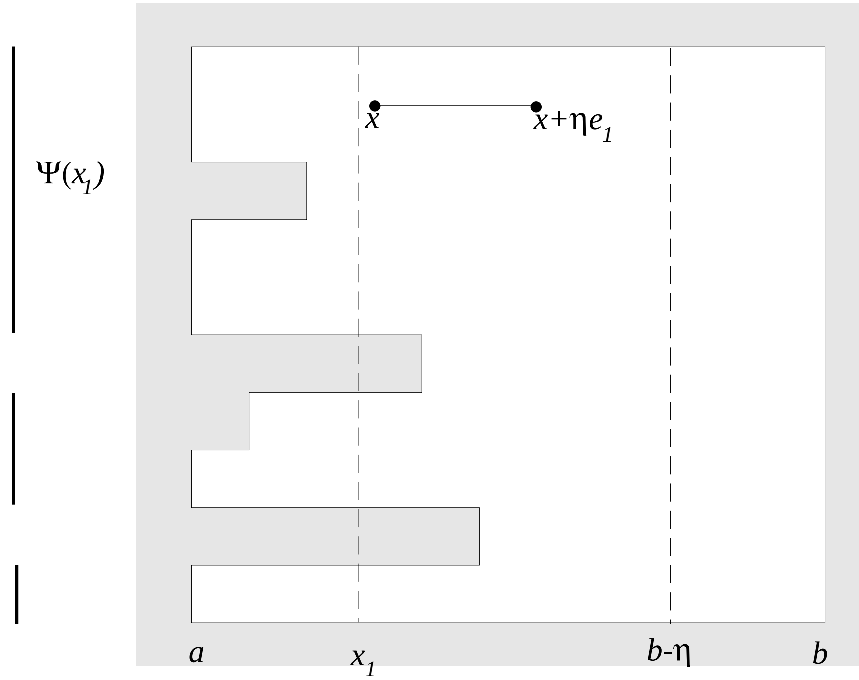

FIGURE 1. A basic interior segment domain in the plane.

Theorem 1. *Let $\Omega \subset (a, b) \times \Psi \subset \mathbb{R}^n$ be a basic interior segment domain for $n > 1$ and assume that Ψ and each component of each open set*

$$\Psi(x_1) = \{x \in \mathbb{R}^{n-1} \mid (x_1, x) \in \Omega\} \qquad a < x_1 < b$$

is a uniform domain in $\mathbb{R}^{n-1}$. Then $C_b^\infty(\Omega)$ is dense in $W^{k,p}(\Omega)$.

Lemma 2.1. *Let Ω_1 and Ω_2 be uniform domains with finite diameters $d(\Omega_1)$ and $d(\Omega_2)$. Then the product $\Omega_1 \times \Omega_2$ is a uniform domain.*

Proof. Let (x_1, x_2) and (y_1, y_2) be points in $\Omega_1 \times \Omega_2$, and assume without loss of generality that $|x_1 - y_1| \leq |x_2 - y_2|$. Choose a point $z_1 \in \Omega_1$ such that

$$|x_1 - z_1| = \frac{|x_2 - y_2| d(\Omega_1)}{2d(\Omega_2)}.$$

By considering points on a curve from x_1 to z_1 satisfying (1.1) and (1.2), we can find $w_1 \in \Omega_1$ such that

$$|x_1 - w_1| = \frac{|x_2 - y_2| d(\Omega_1)}{4d(\Omega_2)}$$

and the distance from w_1 to $\partial\Omega_1$ is at least $|x_1 - w_1|/M_{\Omega_1}$.

Let γ_1 be the curve in Ω_1 from x_1 to y_1 formed by joining curves from x_1 to w_1 and from w_1 to y_1 that satisfy (1.1) and (1.2), and let γ_2 be the curve in Ω_2

from x_2 to y_2 satisfying (1.1) and (1.2). Now let $z_i(t)$, $0 \le t \le L_i$, be the arclength parameterizations of γ_i with $z_i(0) = x_i$, $i = 1, 2$. It follows that $C_0 = L_2/L_1$ is bounded from above and away from 0 by constants depending only on $d(\Omega_i)$ and M_{Ω_i}, $i = 1, 2$. Consider the curve σ given by $(z_1(t), z_2(C_0 t))$, $0 \le t \le L_1$, from (x_1, x_2) to (y_1, y_2) in $\Omega_1 \times \Omega_2$. The arclength of the part of σ corresponding to $a \le t \le b$ is comparable to $b - a$, and the distance from $(z_1(t), z_2(C_0 t))$ to $\partial(\Omega_1 \times \Omega_2)$ is comparable to the minimum of the distances from z_i to $\partial\Omega_i$. Thus, the fact that each γ_i satisfies (1.2) implies that σ also satisfies this condition. Lastly, the length of σ is comparable to the sum of the lengths of the γ_i's, and hence to $|x_2 - y_2|$. It follows that σ satisfies (1.1) and that $\Omega_1 \times \Omega_2$ is a uniform domain.

Remark. By letting Ω_1 be an interval and Ω_2 a half plane, we see that it is necessary to assume that the domains are bounded.

Observe that if $\Omega \subset (a, b) \times \Psi$ is a basic interior segment domain, then there is an upper semicontinuous (usc) function $f : \Psi \to \mathbb{R}$ such that for each $x' \in \Psi$:

$$(x_1, x') \in \Omega \quad \text{if and only if} \quad f(x') < x_1 < b.$$

We say that f is the defining function for Ω. To make the proof of Theorem 1 easier to follow, we first prove a special case:

Lemma 2.2. *Suppose $\Omega \subset (a, b) \times \Psi$ satisfies the hypotheses of Theorem 1, and further assume that the defining function for Ω, $f : \Psi \to \mathbb{R}$, assumes only finitely many values. Then $C_b^\infty(\Omega)$ is dense in $W^{k,p}(\Omega)$.*

Proof. The proof is by induction on the number of values f assumes. First consider the case that f is constant. Then, $\Omega = J \times \Psi$ for some interval J. Since intervals are uniform domains, Lemma 2.1 implies Ω is a uniform domain and we are done because, as noted before, uniform domains have Property 1.

Now suppose that the defining function f has m values, and let $u \in W^{k,p}(\Omega)$. Let a_m be the largest value, so that $\Omega^0 = (a_m, b) \times \Psi$ is the part of Ω with $x_1 > a_m$. Since Ω^0 is a uniform domain, there is a global Sobolev function U whose restriction to Ω^0 is u. By subtracting U, which clearly can be approximated by functions in $C_b^\infty(\Omega)$, we see that we may assume u is identically zero on Ω^0.

We decompose Ω into the disjoint union

$$\Omega = (\Omega \cap \overline{\Omega^0}) \cup \Omega^1 \cup \Omega^2 \dots ,$$

where the Ω^k's, for $k \ge 1$, are the domains determined by the components of the open set $\Psi(a_m)$ in $\mathbb{R}^{n-1}$. By hypothesis, each component of each cross-section is a uniform domain and furthermore, the defining function for each Ω^k, with $k > 0$, has fewer than m values. Since $\cup\Omega^k$ has finite volume, there is an integer K such that we may redefine u to be identically zero on Ω^k for all $k > K$ and still have the modified function v within ϵ of u in the Sobolev norm.

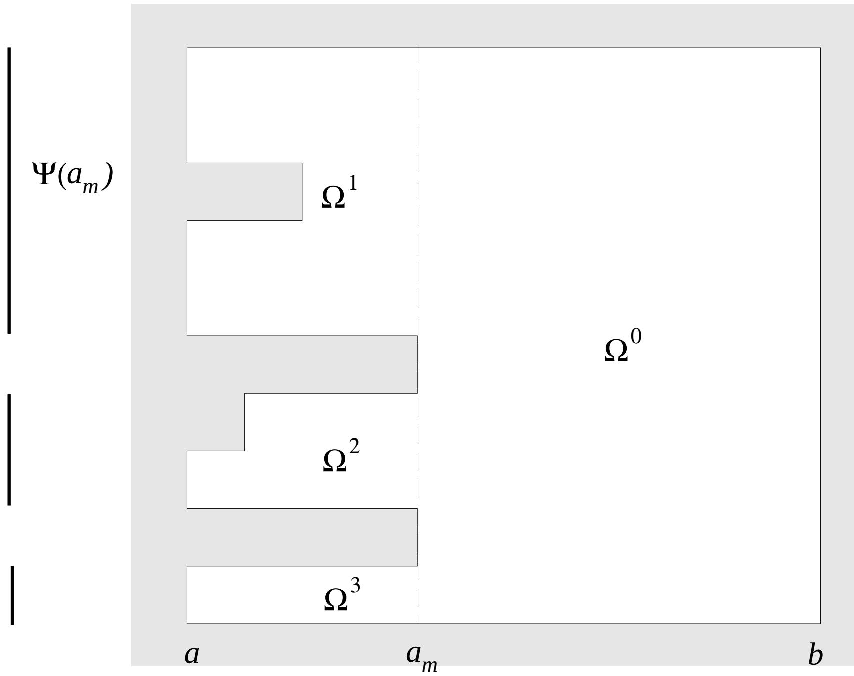

FIGURE 2. Decomposing Ω in order to apply induction.

Since v is 0 on Ω^0, for $0 < \lambda < b - a_m$ we can define $v_\lambda(x) = v(x + \lambda e_1)$ on Ω, by extending v to be 0 for $x_1 \geq b$. By the L^p-continuity of translation, we can choose λ sufficiently small so that v_λ is within ϵ of v in Sobolev norm. Note that there is a neighborhood of a_m such that v_λ is 0 whenever x_1 is in that neighborhood.

By induction, we can approximate the function v_λ restricted to Ω^k by a function $w_k \in C_b^\infty(\Omega^k)$ which differs from v_λ in the Sobolev norm by an amount that is less than ϵ/K. Moreover, we may assume that each w_k vanishes for x_1 in a neighborhood of a_m, since v_λ is 0 there. It follows that the function w which is zero on $\Omega \setminus (\Omega^1 \cup \cdots \cup \Omega^K)$ and $w = w_k$ on Ω^k, $1 \leq k \leq K$, is a well defined function in $C_b^\infty(\Omega)$ which differs from u in the Sobolev norm by an amount that is less than $C\epsilon$. This completes the proof.

We now ready to complete the proof of Theorem 1.

Proof of Theorem 1. Let $u \in W^{k,p}(\Omega)$ and $\epsilon > 0$. We may assume that $u \in C^\infty(\Omega)$ by the Meyers-Serrin Theorem. Put $\lambda_0 = \eta/2$ and let $\phi \in C^\infty(\mathbb{R})$ satisfy $\phi = 0$ for $x_1 \leq b - \eta$ and $\phi = 1$ for $x_1 \geq b - \lambda_0$. Write $u = u_1 + u_2$, where $u_1(x) = \phi(x_1)u(x)$ for $x_1 > b - \eta$ and $u_1(x) = 0$ for $x_1 \leq b - \eta$. Now, we see that $u_1 \in W^{k,p}((a, b) \times \Psi)$ and hence by Lemma 2.1, Jones' Extension Theorem and Meyers-Serrin approximation it follows that u_1 can be approximated by functions in $C_b^\infty(\Omega)$. Thus, it suffices to approximate u_2. But $u_2 = (1 - \phi)u = 0$ for all $x_1 \geq b - \lambda_0$. We may therefore assume, without loss of generality, that our *original* function u satisfies $u = 0$ for all $b > x_1 > b - \lambda_0$.

For $0 < \lambda < \lambda_0$, we can now define $u_\lambda(x) = u(x + \lambda e_1)$, by extending u to be zero for $x_1 > b - \lambda_0$, and get a $W^{k,p}$ function on the shifted domain $(\Omega - \lambda e_1) \cup \Omega$ with norm the same as the norm of u on Ω. By the L^p-continuity of translation we again get that there exists $\lambda_1 > 0$ with

$$\|u - u_{\lambda_1}\|_{W^{k,p}(\Omega)} < \epsilon. \tag{2.1}$$

Let $\Omega_1 = (\Omega - \lambda_1 e_1) \cup \Omega$ and suppose that there is a domain Ω' with $\Omega \subset \Omega' \subset \Omega_1$ with the C_b^∞ approximation property. Then we could find $v \in C_b^\infty(\Omega')$ such that

$$\|u_{\lambda_1} - v\|_{W^{k,p}(\Omega')} < \epsilon.$$

It would follow that

$$\|u - v\|_{W^{k,p}(\Omega)} \leq \|u - u_{\lambda_1}\|_{W^{k,p}(\Omega)} + \|u_{\lambda_1} - v\|_{W^{k,p}(\Omega')} < 2\epsilon\,,$$

and hence u could be approximated by a function in $C_b^\infty(\Omega)$. Hence the theorem will be proved once such a domain is constructed.

Let $f : \Psi \to \mathbb{R}$ be the defining function for Ω. We define Ω' by

$$\Omega' = \{(x_1, x') \mid x' \in \Psi \text{ and } \left[\frac{f(x')}{\lambda_1}\right]\lambda_1 < x_1 < b\}.$$

Here $[t]$ denotes the greatest integer less than or equal to t. Clearly $\Omega \subset \Omega' \subset \Omega_1$, and since f is usc it follows that Ω' is an open set.

If we let $f' : \Psi \to \mathbb{R}$ be the corresponding usc function which defines Ω' then we see that f' takes on only *finitely* many values. Therefore Lemma 2.2 allows us to conclude that Ω' has the required C_b^∞ approximation property, and this completes the proof.

3. A Smooth Cantor-Type Sneak Function

The Cantor function is a function $\phi : [0,1] \to \mathbb{R}$ which is continuous, satisfies $\phi' = 0$ a.e. and yet ϕ manages to sneak from the value zero to the value one. We construct a similar function which is smooth and does most of its increasing on a Cantor set of positive measure. We note that the desired function in the theorem can not be constructed by the usual mollification of the characteristic function of K.

Theorem 2. *Let K be a closed subset of the open interval $I = (a,b)$. Assume that K has positive measure. If $\epsilon > 0$, $k \in \{1,2,\dots\}$, and $1 \leq p < \infty$, then there is a function $\phi \in C^\infty(\mathbb{R})$ satisfying*

$$\phi(x) = 0 \text{ for } x \leq a,\ \phi(x) = 1 \text{ for } x \geq b \tag{3.1}$$

and

$$\int\limits_{I \setminus K} |\phi^{(j)}|^p\, dx < \epsilon \qquad j = 1, \dots, k\,. \tag{3.2}$$

Proof. First consider the case $k = 1$. Denote the measure of a set E by $|E|$. There is an open set U with $K \subset U \subset I$ and $|U \setminus K| < |K|^p \epsilon$. Let $\psi \in C^\infty(\mathbb{R})$ satisfy $0 \leq \psi \leq 1$, $\psi = 1$ on K and $\psi = 0$ on U^c. Put

$$\phi(x) = \begin{cases} 0 & \text{if } x \leq a \\ \int_a^x \left[\dfrac{\psi(t)}{\int\limits_I \psi\, dt}\right] dt & \text{if } x > a\,. \end{cases} \tag{3.3}$$

so that ϕ clearly satisfies (3.1). Moreover,

$$\begin{aligned}\int\limits_{I\setminus K} |\phi'|^p\,dx &= \int\limits_{I\setminus K} \left[\frac{\psi(t)}{\int_I \psi\,dt}\right]^p dt\\ &\le \frac{1}{|K|^p}\int\limits_{U\setminus K} \psi^p\,dt\\ &\le \frac{|U\setminus K|}{|K|^p} < \epsilon\,,\end{aligned}$$

which proves (3.2).

Now assume, by induction, that the theorem has been proved for some $k \ge 1$. There exists an open set $U \subset I$ containing K with $2^p|U\setminus K| < \epsilon|K|^p$. Since K is compact, U has n interval components J_i, $i = 1,\dots,n$, which cover K and without loss of generality such that $K_i = K\cap J_i$ has positive measure.

We now apply the induction hypothesis to appropriate subintervals of J_i to produce a function that first goes from 0 to 1 on an interval $J_{i,1} \subset J_i$ satisfying $|J_{i,1}\cap K| = |K_i|/4$, remains constant on an interval $J_{i,2}$ satisfying $|J_{i,2}\cap K| = |K_i|/2$, and then decreases from 1 to 0 on an interval $J_{i,3}$ satisfying $|J_{i,3}\cap K| = |K_i|/4$. Thus, we can find $\psi \in C^\infty(\mathbb{R})$ such that

$$0 \le \psi \le 1 \text{ and } \psi = 0 \text{ on } U^c, \tag{3.4}$$

$$\int\limits_{J_i} \psi\,dx \ge \int\limits_{J_{i,2}} \psi\,dx \ge \frac{1}{2}|K_i|\,, \qquad i = 1,\dots,n \tag{3.5}$$

and

$$\int\limits_{I\setminus K} |\psi^{(j)}|^p\,dx < \frac{\epsilon|K|^p}{2^p} \qquad j = 1,\dots,k. \tag{3.6}$$

The function ψ is a "bump function" over each K_i on J_i. Define ϕ by (3.3), so that again $\phi \in C^\infty(\mathbb{R})$ and satisfies (3.1).

Now, for $2 \le j \le k+1$,

$$\begin{aligned}\int\limits_{I\setminus K} |\phi^{(j)}|^p\,dx &= \int\limits_{I\setminus K} \left|\frac{\psi^{(j-1)}(t)}{\int_I \psi\,dt}\right|^p dt\\ &\le \frac{2^p}{|K|^p}\int\limits_{I\setminus K} |\psi^{(j-1)}(t)|^p\,dt\\ &< \epsilon\,,\end{aligned}$$

by (3.5) and (3.6), and for $j = 1$,

$$\int\limits_{I\setminus K} |\phi'|^p\,dx \le \frac{2^p}{|K|^p}\int\limits_{U\setminus K} |\psi|^p\,dx \le \frac{2^p}{|K|^p}|U\setminus K| < \epsilon\,,$$

by (3.4) and (3.5). Thus, (3.2) is satisfied and the proof is complete.

4. Approximation by Global C^∞ Functions

Let Ω be a basic interior segment domain as in Section 2, except we now restrict our attention to domains in the plane. Recall that a boundary point $z \in \partial\Omega$ is a m_2-limit point of Ω^c if the area of $B(z,r)\backslash\Omega$ is positive for all disks $B(z,r)$ centered at z and of radius $r > 0$.

Theorem 3. *Let $\Omega \subset \mathbb{R}^2$ be a basic interior segment domain. A sufficient condition for $C^\infty(\overline{\Omega})$ to be dense in $W^{k,p}(\Omega)$ is that every boundary point of Ω be a m_2-limit point of Ω^c.*

Lemma 4.1. *Let $R = (0,a) \times (0,b)$ be a rectangle and suppose that $f \in C^{k+1}(R)$ with $|D^\alpha f(z)| \le M$ for all $z \in R$ and $|\alpha| \le k+1$. Denote by $\tilde{R}$ the rectangle $(0,a) \times (0,2b)$. Then, there is a function $F \in C^k(\tilde{R})$ with $F = f$ on R and*

$$|D^\alpha F(z)| \le CM, \quad z \in \tilde{R}, \quad |\alpha| \le k$$

where C is some absolute constant.

Proof of the lemma. See the proof of Lemma 6.37 in [5], page 136.

Proof of Theorem 3. Let Ω be a basic interior segment domain with boundary satisfying the above density condition. Using the same techniques as in the proof of Theorem 1, we see that it suffices to consider a domain Ω whose defining function assumes only m values. Let a_m be the maximum value of f which is now defined on some interval Ψ. Let $u \in W^{k,p}(\Omega)$ and let $\epsilon > 0$. Again, as in the proof of Theorem 1, we may assume that $u = 0$ on the strip $(a_m, b) \times \Psi$ and in fact, we may assume that $u = 0$ if $x > a_m - \lambda$ for some $\lambda > 0$.

Decompose Ω into the disjoint union $\Omega = (\Omega \cap \overline{\Omega^0}) \cup \Omega^1 \cup \ldots$ as in the proof of Theorem 1. The Ω^j's are determined by the interval components of the open set $\Psi(a_m)$ in $\mathbb{R}^1$. Since $\sum m_2(\Omega^j)$ converges (where m_2 is two dimensional Lebesgue measure), there is an integer J such that we may redefine u to be identically zero on $\Omega^{J+1} \cup \ldots$ and still have the modified function u_1 within ϵ of u in the $W^{k,p}(\Omega)$ norm.

By Theorem 1, applied to each Ω^j, there is a function $v \in C_b^\infty(\Omega)$ which is within ϵ of u_1 in the Sobolev norm and vanishes on all of Ω except possibly on the Ω^j's. Also, $v = 0$ for all $x > a_m - \lambda/2$. Put

$$M = \sup_{z \in \Omega, |\alpha| \le k} |D^\alpha v|,$$

which is finite since $v \in C_b^\infty(\Omega)$.

Let a_{m-1} be the second largest value for the defining function f. For each $j = 1, \ldots, J$, the intersection $\Omega^j \cap \{(x,y) \in \mathbb{R}^2 : a_{m-1} < x < a_m\}$ is a rectangle $(a_{m-1}, a_m) \times I^j$, where the intervals I^j are components of $\Psi(a_m)$. Now of course the intervals $\{I^j\}$ are disjoint by the construction, but moreover, the density condition implies that their closures are disjoint. Let $I^j = (c_j, d_j)$ and assume that the intervals are ordered so that $d_{j-1} < c_j$.

For $d_{j-1} < y < c_j$, observe that either the point $(a_m, y) \in \partial\Omega$ or else the segment $(a_{m-1}, b) \times \{y\}$ is in Ω. Consequently, $\partial\Omega \cap [\{a_m\} \times (d_j, d_j + \eta)]$ and $\partial\Omega \cap [\{a_m\} \times (c_j - \eta, c_j)]$ must both have positive linear measure by the density condition, for *any* $\eta > 0$. We now extend the definition of v to the whole strip

$S = (a_{m-1}, b) \times \Psi$ using Theorem 2. Fix η, with $0 < \eta < \frac{1}{2}\min\{c_j - d_{j-1}\}$ and $\eta < \epsilon/(JM^p(a_m - a_{m-1}))$. Now apply Theorem 2 in each of the intervals $(c_j - \eta, c_j)$, $(d_j, d_j + \eta)$ to construct a function $\psi \in C^\infty(\mathbb{R}^1)$ with $0 \le \psi \le 1$ and such that $\psi = 1$ on $\cup(c_j, d_j)$, $\psi = 0$ off $\cup(c_j - \eta, d_j + \eta)$ and

$$\int_{K^c} |\psi^{(l)}|^p \, dx < \frac{\epsilon}{JM^p}, \qquad l = 1, \ldots, k,$$

where $K = \partial\Omega \cap \{x = a_m\}$.

By Lemma 4.1 there is a function v_1 which is C^∞ on the set

$$G = \bigcup_{j=1}^{J} (a_{m-1}, a_m) \times (c_j - \eta, d_j + \eta)$$

and whose restriction to $\cup_{j=1}^{J}(a_{m-1}, a_m) \times (c_j, d_j)$ is v. Moreover, $|D^\alpha v_1| \le CM$ for all $|\alpha| \le k$.

Now put $v_2(x, y) = v_1(x, y)\psi(y)$ for (x, y) in $\Omega \cup G$ and $v_2 = 0$ on $S \setminus G$. Since $v = 0$ for all $x > a_m - \lambda/2$ we see that v_2 is a C^∞-function on $S \cup \Omega$. Finally, we observe that

$$\begin{aligned}
\|v - v_2\|_{W^{k,p}(\Omega)} &= \|v_2\|_{W^{k,p}(\Omega \cap G \setminus (\Omega^1 \cup \cdots \cup \Omega^J))} \\
&= \|v_1\psi\|_{W^{k,p}(\Omega \cap G \setminus (\Omega^1 \cup \cdots \cup \Omega^J))} \\
&= \sum_{j=1}^{J} \|v_1\psi\|_{W^{k,p}([(a_{m-1}, a_m) \times (c_j - \eta, c_j)] \cap \Omega)} \\
&\quad + \sum_{j=1}^{J} \|v_1\psi\|_{W^{k,p}([(a_{m-1}, a_m) \times (d_j, d_j + \eta)] \cap \Omega)},
\end{aligned}$$

and we will show this quantity is small due to the choice of η.

Consider a typical term in the derivative of the product $v_1\psi$. We estimate as follows:

$$\begin{aligned}
&\int_{a_{m-1}}^{a_m} \int_{(c_j - \eta, c_j) \setminus K} |D^\alpha v_1 \psi^{(l)}(y)|^p \, dy dx \\
&\qquad \le CM^p(a_m - a_{m-1}) \int_{K^c} |\psi^{(l)}(y)|^p \, dy dx \\
&\qquad \le CM^p \frac{\epsilon}{JM^p} = \frac{C\epsilon}{J},
\end{aligned}$$

for all $l > 0$. Similarly, for $l = 0$, we get

$$\int_{a_{m-1}}^{a_m} \int_{(c_j - \eta, c_j) \setminus K} |D^\alpha v_1 \psi(y)|^p \, dy dx \le CM^p(a_m - a_{m-1})\eta < \frac{C\epsilon}{J},$$

by the choice of η. The same estimate applies when the integration is done over $(a_{m-1}, a_m) \times (d_j, d_j + \eta)$. Adding these estimates, we see that $\|v - v_2\|_{W^{k,p}(\Omega)}$ is smaller than a constant (depending on k) times ϵ.

This proves that a Sobolev function on Ω can be approximated by a function $w \in C^\infty(\Omega \cup S)$. Consider such a function w. As argued before, we may as well assume that $w = 0$ on S, because S is a uniform domain. We may also assume that $w = 0$ for x in a neighborhood of a_{m-1}. By induction, $w_j = w\,|_{\Omega^j}$ can be approximated, within ϵ/J in the Sobolev norm, by a global C^∞-function which also vanishes for $x \geq a_{m-1}$ and such that $w_j\,|_{\Omega^k} \equiv 0$ for $j \neq k$. Adding these functions gives a function in $C^\infty(\overline{\Omega})$ which is within ϵ of u in the $W^{k,p}(\Omega)$ norm. This completes the proof.

5. Approximation on Starshaped and Interior Segment Domains

In this section we prove Theorems A and B of the introduction. Our method is based on the following principle. First, there are general homeomorphisms which preserve the density conditions of Properties 1 and 2 (of the introduction) as well as Lebesgue density points. For example, it is clear that bilipschitz maps enjoy these properties, more generally, so do the so-called quasi-isometric maps which are studied in Chapter 1 of [10]. By invoking such homeomorphisms in conjunction with standard partition of unity arguments, we can obtain results similar to those of Theorems 1 and 3 for different sorts of domains.

Theorem 4. *Let Ω be a bounded domain in $\mathbb{R}^2$. If Ω satisfies the interior segment condition, then $C_b^\infty(\Omega)$ is dense in $W^{k,p}(\Omega)$ for all $k = 1, 2, \ldots$ and all p, $1 \leq p < \infty$.*

Proof. For each $z \in \partial\Omega$ let $\eta_z > 0$ and $y_z \neq 0$ be as in the definition of the interior segment condition. Choose points $z_1, z_2, \ldots, z_N \in \partial\Omega$ so that the balls $\{B(z_i, \eta_{z_i}/2)\}_{i=1}^N$ cover the compact set $\partial\Omega$.

We now fix an index i, $1 \leq i \leq N$, and introduce new (orthogonal) coordinates (x_1, x_2) so that y_{z_i} lies on the x_1-axis. Next, replacing the given cover by a cover of smaller balls if necessary, we may assume that for some rectangle

$$R_i = \{\, (x_1, x_2) \mid a_i < x_1 < b_i \text{ and } c_i < x_2 < d_i \,\}$$

we have

$$\Omega \cap B(z_i, \eta_{z_i}/2) \subset R_i$$

and $\Omega_i = \Omega \cap R_i$ is a basic interior segment domain. Finally, choose a domain R_0 containing the closure of $\Omega \setminus \cup R_i$ and which is itself compactly contained in Ω. We now select a partition of unity $\{\lambda_i\}_{i=0}^N$ of Ω subordinate to the covering $\{R_0, R_1, \ldots, R_N\}$. This means that there are functions $\lambda_i \in C^\infty$ for which

$$\lambda_i \geq 0, \qquad \lambda_i \equiv 0 \quad \text{on } R_i^c, \qquad \text{and} \qquad \sum \lambda_i \equiv 1 \quad \text{on } \Omega.$$

With this we are ready to approximate. So assume that $u \in W^{k,p}(\Omega)$ and $\epsilon > 0$ are given. As before, the Meyers-Serrin theorem permits us to assume that $u \in C^\infty(\Omega)$. For each $i > 0$, Theorem 1 provides us with a function $f_i \in C_b^\infty(\Omega_i)$ with

$$\| f_i - u_i \|_{W^{k,p}(\Omega_i)} < \frac{\epsilon}{N}.$$

(We use the notation u_i to denote the restriction of the function u to the Ω_i.) Letting $f_0 = u$ and extending f_i to be 0 off Ω_i for $i > 0$, we consider the function $f = \sum f_i \lambda_i$. Certainly $f \in C_b^\infty(\Omega)$ and since $\sum \lambda_i \equiv 1$ on Ω, we have that

$$\begin{aligned} \| f - u \|_{W^{k,p}(\Omega)} &\leq \sum \| \lambda_i (f-u) \|_{W^{k,p}(\Omega_i)} \\ &\leq \mathrm{C} \sum_{i>0} \| f - u \|_{W^{k,p}(\Omega_i)} \\ &< \mathrm{C}\epsilon. \end{aligned}$$

Here C is a constant depending only on k, p, and the λ_i's; the second inequality results from the Leibniz rule. The proof is thus complete.

Remark. *We observe that in the application of Theorem 1 in the above proof, it was automatically true that the basic interior segment domains Ω_i satisfied the uniform domain requirements because, the components of $\Psi(x_1)$ are just intervals in $\mathbb{R}^1$. The situation for $n > 2$ is more complicated and in fact the theorem is no longer true; see Section 7. The situation is similar in the next three theorems.*

Theorem 5. *Let Ω be a bounded domain in $\mathbb{R}^2$ which satisfies the interior segment condition. If every $z \in \partial\Omega$ is an m_2-limit point of Ω^c, then $C^\infty(\overline{\Omega})$ is dense in $W^{k,p}(\Omega)$ for all $k = 1, 2, \ldots$ and all p, $1 \leq p < \infty$.*

Proof. The proof is the same as that of Theorem 4 except for the use of Theorem 3 in place of Theorem 1.

Theorem 6. *Let Ω be a bounded starshaped domain in $\mathbb{R}^2$. Then $C_b^\infty(\Omega)$ is dense in $W^{k,p}(\Omega)$ for all $k = 1, 2, \ldots$ and all p, $1 \leq p < \infty$.*

Proof. For convenience, we assume that Ω is starshaped with respect to the origin. Choose numbers r and R, $0 < r < R < \infty$, such that

$$\partial\Omega \subset \{ z \mid r < |z| < R \}.$$

We next cover Ω with the following three open sets:

$$\begin{aligned} V_1 &= \{ z : r/2 < |z| < R, \arg\{z/|z|\} \in (\pi/4, 7\pi/4) \}, \\ V_2 &= \{ z : r/2 < |z| < R, \arg\{z/|z|\} \in (-3\pi/4, 3\pi/4) \} \quad \text{and} \\ V_3 &= \{ z : |z| < r \}. \end{aligned}$$

Suitable modifications of the exponential function provide us with conformal maps $\Phi_i : R_i \to V_i$, $i = 1, 2$, where R_i is a rectangle. Note that these mappings Φ_i have the property that (for a fixed k) all of the partial derivatives of Φ_i and of its inverse Φ_i^{-1} of order at most k are uniformly bounded. Letting

$$\Omega_i = \Omega \cap V_i, \quad (i = 1, 2)$$

we conclude there exists a constant C such that

$$\frac{1}{\mathrm{C}} \| u \circ \Phi_i \|_{W^{k,p}(\Phi_i(\Omega_i))} \leq \| u \|_{W^{k,p}(\Omega_i)} \leq \mathrm{C} \| u \circ \Phi_i \|_{W^{k,p}(\Omega_i)}.$$

Suppose now we are given a function $u \in W^{k,p}(\Omega) \cap C^\infty(\Omega)$ along with a number $\epsilon > 0$. For $i = 1, 2$, it is clear that $\Phi_i(\Omega_i)$ is a basic interior segment domain, hence by invoking Theorem 1 we obtain functions $f_i \in C_b^\infty(\Phi_i(\Omega_i))$ such that

$$\| f_i - u_i \circ \Phi_i \|_{W^{k,p}(\Phi_i(\Omega_i))} < \epsilon.$$

Hence,

$$\| f_i \circ \Phi_i^{-1} - u_i \|_{W^{k,p}(\Omega_i)} < \mathrm{C}\epsilon.$$

If we select a partition of unity $\{\lambda_1, \lambda_2, \lambda_3\}$ subordinate to the open cover $\{ V_1, V_2, V_3 \}$ of Ω then, just as in the proof of Theorem 4, one can show that the function

$$f = f_1 \circ \Phi_1^{-1} \lambda_1 + f_2 \circ \Phi_2^{-1} \lambda_2 + u \lambda_3$$

is in $C_b^\infty(\Omega)$ and satisfies

$$\| f - u \|_{W^{k,p}(\Omega)} < \mathrm{C}\epsilon.$$

This completes the proof of Theorem 6 and as before we also have the following theorem:

Theorem 7. *Let Ω be a bounded starshaped domain in $\mathbb{R}^2$. If every $z \in \partial\Omega$ is a m_2-limit point of Ω^c, then $C^\infty(\overline{\Omega})$ is dense in $W^{k,p}(\Omega)$ for all $k = 1, 2, \ldots$ and all p, $1 \le p < \infty$.*

We end this section with another geometric sufficient condition for globally smooth approximation. This theorem will, under certain circumstances, allow us to ignore finitely many "bad points" of a domain in which we want to approximate Sobolev functions. We will make use of it in some of our constructions of examples in Section 7. To facilitate the statement and proof of this result, we begin with a definition and a lemma.

Definition. *Suppose that we are given a bounded domain $\Omega \in \mathbb{R}^2$, with distinct points $\zeta_1, \zeta_2, \ldots, \zeta_N \in \partial\Omega$, and a positive number $\delta < \min\{ |\zeta_i - \zeta_j|/2 \mid i \ne j \}$. We then define the δ-modification of Ω with respect to $\{ \zeta_1, \zeta_2, \ldots, \zeta_N \}$ as*

$$\Omega(\delta; \zeta_1, \zeta_2, \ldots, \zeta_N) = \Omega \setminus \bigcup_{j=1}^{N} \overline{B(\zeta_j, \delta)}.$$

Lemma 5.1. *Suppose that $1 \le p \le 2$ and that ϵ and δ are positive numbers. Then there exist numbers δ_1 and δ_2 with $0 < \delta_1 < \delta_2 < \delta$ and a function $\phi \in C^\infty(\mathbb{R}^2)$, $0 \le \phi \le 1$, satisfying*

(i) *$\phi \equiv 0$ on $B(0, \delta_1)$,*
(ii) *$\phi \equiv 1$ off $B(0, \delta_2)$, and*
(iii) *$\iint_{\mathbb{R}^2} |\nabla \phi|^p \, dxdy < \epsilon$.*

Proof. Case 1: $1 \le p < 2$. Choose the positive number δ_1 to satisfy

$$\delta_1 < \min\left\{ \sqrt[2-p]{\epsilon/3\pi}, \delta/2 \right\}.$$

Next, put $\delta_2 = 2\delta_1$ and consider the Lipschitz radial function $\phi_0 : \mathbb{R}^2 \to \mathbb{R}^1$ defined by

$$\phi_0(x) = \begin{cases} 0 & \text{if } |x| < \delta_1 \\ 1 & \text{if } |x| > \delta_2 \text{ and} \\ (|x| - \delta_1)/\delta_1 & \text{otherwise.} \end{cases}$$

Integration in polar coordinates yields

$$\iint_{\mathbb{R}^2} |\nabla \phi_0|^p \, dxdy = 2\pi \int_{\delta_1}^{\delta_2} \frac{r}{\delta_1^p} \, dr = 3\pi \delta_1^{2-p} < \epsilon.$$

Taking ϕ to be a suitable mollification of ϕ_0, we get a smooth function ϕ satisfying (iii) which equals 1 off $B(0, \delta)$ and vanishes in some neighborhood of the origin. By appropriately decreasing δ_1 and increasing δ_2 we see that this ϕ has all of the desired properties.

Case 2: $p = 2$. Let δ_2 be any positive number less than δ and choose δ_1 to be another positive number satisfying

$$\delta_1 \exp \frac{2\pi}{\epsilon} < \delta_2.$$

Next we consider the Lipschitz radial function $\phi_0 : \mathbb{R}^2 \to \mathbb{R}^1$ defined by

$$\phi_0(x) = \begin{cases} 0 & \text{if } |x| < \delta_1 \\ 1 & \text{if } |x| > \delta_2 \text{ and} \\ \dfrac{\log(\delta_1/|x|)}{\log(\delta_1/\delta_2)} & \text{otherwise.} \end{cases}$$

The proof can be then completed as before, since an integration using polar coordinates again yields $\iint_{\mathbb{R}^2} |\nabla \phi_0|^2 \, dxdy < \epsilon$.

Theorem 8. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, let $1 \le p \le 2$, and let $\zeta_1, \zeta_2, \ldots, \zeta_N$ be distinct points of $\partial\Omega$. If there exists a null sequence of positive numbers, $\{\delta_i\}_{i=1}^{\infty}$, such that for each i, the boundary of $\Omega(\delta_i; \zeta_1, \zeta_2, \ldots, \zeta_N)$ consists of finitely many disjoint Jordan curves, then $C^\infty(\mathbb{R}^2)$ is dense in $W^{1,p}(\Omega)$.*

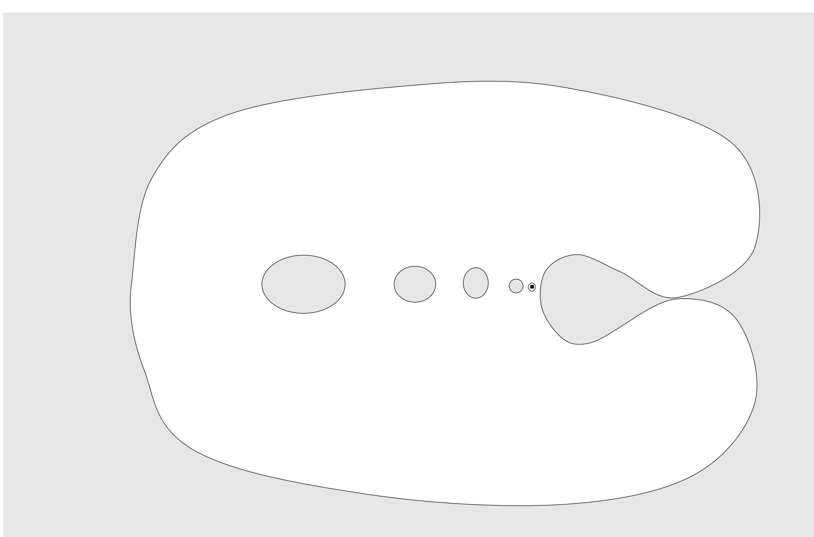

FIGURE 3. A domain to which Theorem 8 is applicable.

Proof. For general domains (in $\mathbb{R}^n$) it is known that $W^{1,p}(\Omega) \cap C^\infty(\Omega) \cap L^\infty(\Omega)$ is dense in $W^{1,p}(\Omega)$ for all p, $1 \leq p < \infty$; see Corollary 3.1.2 on page 162 in [10]. Thus, we need only show that we can approximate Sobolev functions which are bounded and smooth. We now fix an exponent p, $1 \leq p \leq 2$, along with a smooth bounded function u in the corresponding Sobolev space.

For each ζ_j, $1 \leq j \leq N$, $\epsilon > 0$ and $\delta > 0$, we let $\phi_{\zeta_j,\epsilon,\delta}$ denote the function $\phi(x - \zeta_j)$, where ϕ is the function satisfying the properties (i)–(iii) of Lemma 5.1. Observe that (by using the product rule for differentiation) the function

$$v = u \prod_{j=1}^{N} \phi_{\zeta_j,\epsilon,\delta}$$

can be made as close as we wish to u in $W^{1,p}(\Omega)-$norm by choosing ϵ and δ appropriately small. Also, for a sufficiently large index i, we observe that v vanishes on $\cup_{j=1}^{N} B(\zeta_j, \delta_i) \cap \Omega$. Now by assumption, the domain $\Omega(\delta_i; \zeta_1, \zeta_2, \ldots, \zeta_N)$ is a finitely connected domain whose boundary consists of finitely many disjoint Jordan curves $\{\Gamma_l\}$. One of these boundary components, call it Γ_0, will contain $\Omega(\delta_i; \zeta_1, \zeta_2, \ldots, \zeta_N)$ in its interior, while the others will contain this domain in their exteriors. We appeal to a result of Lewis [9], which says that for any (bounded) Jordan domain D in the plane, $C^\infty(\mathbb{R}^2)$ is dense in $W^{1,p}(D)$ for any p, $1 \leq p < \infty$. The proof can now be completed by a partition of unity argument similar to the ones used above. For the truncation of v to an $\Omega-$neighborhood of Γ_0, we use the Lewis result directly to approximate on the interior of Γ_0. For the trunctation of v to an $\Omega-$neighborhood of Γ_l, $l \neq 0$, we again use the Lewis result, only after a preliminary (complex analytic) inversion with respect to any point in the interior of Γ_l.

6. NECESSARY CONDITIONS

In this section we establish some necessary conditions for approximating Sobolev functions. We begin by showing that it is necessary for every limit point of the set of nondegenerate components of $\partial\Omega$ to be a m_2-limit point of Ω^c in order to have $C^\infty(\overline{\Omega})$ dense in $W^{k,p}(\Omega)$. The following result concerning continua in the plane will be needed.

Proposition 6.1. *Let K be a compact connected subset of $\mathbb{R}^2$ containing at least two distinct points and let r_0 be a positive number. Then at least one of the following holds:*

(i) *There exists $a \in K$ and $r \in (0, r_0)$ such that $B(a, r) \setminus K$ has at least two components;*
(ii) *K contains a nontrivial circular arc.*

Remark. The example $K = \overline{B(0,1)}$ shows that (i) may not occur. Also, as the proof will show, if (ii) fails then the set of radii r for which the condition (i) holds will be a dense subset of the interval $[0, (1/2)\text{diam } K]$.

Proof. Since the statements are invariant under translations and dilations, we may assume that $0 \in K$ and that there is a point $z_0 \in K \setminus B(0,2)$. Consider $K_0 = K \cap \partial B(0,1)$. If K_0 contains a nontrivial arc, then (ii) is satisfied and nothing further is required. Thus we assume that K_0 has empty interior relative to the circle $\partial B(0,1)$.

Each $z \in K_0$ is contained in a disk $B(z,\epsilon(z))$ such that $\epsilon(z) < \min\{1/2, r_0\}$ and such that $\partial B(z,\epsilon(z))$ is disjoint from K_0. Since K_0 is compact, it is contained in finitely many such disks, $\{\Delta_1, \ldots, \Delta_n\}$ with the property that no Δ_j is contained in $\cup_{i \neq j} \Delta_i$. If $\Delta_j \setminus K$ has at least two components, for some $1 \leq j \leq n$, then we are done since (i) is satisfied. Suppose to the contrary, that each of the open sets $\Delta_j \setminus K$ is connected, for $1 \leq j \leq n$. We show that this leads to a contradiction.

Assume first that the Δ_j's are disjoint. For each j, $1 \leq j \leq n$, there is a simple curve $\gamma_j \subset \overline{\Delta}_j \setminus K$ that connects the two points of the set $\partial\Delta_j \cap \partial B(0,1)$. Let $\{A_j\}$ be the n subarcs which are the components of the set $\partial B(0,1) \setminus \cup \Delta_j$. Each A_j is a circular arc disjoint from K. Combining the γ_j's with the A_j's we get a closed Jordan curve Γ disjoint from K. See Figure 4 below. Each portion γ_j of Γ can be continuously deformed within Δ_j to the circular arc $\partial B(0,1) \cap \overline{\Delta}_j$ and consequently Γ can be deformed to the unit circle $\partial B(0,1)$ within the annulus $1/2 \leq |z| \leq 3/2$. Hence the winding number of Γ about 0 is one, while the winding number of Γ about z_0 is zero. Whence 0 and z_0 lie in different components of K, but this violates the connectedness of K. This contradiction proves the proposition is the disjoint case. However, the general case is just slightly more difficult and is left to the reader. This completes the proof.

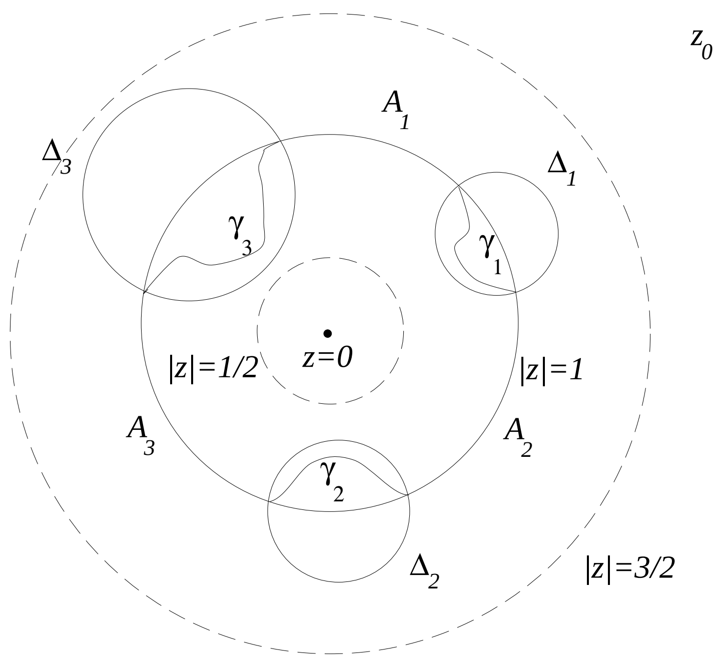

FIGURE 4. Contructing the curve Γ in the proof of Proposition 6.1.

Theorem C. *Let Ω be a domain in $\mathbb{R}^2$. Suppose that $z \in \partial\Omega$, that z is not a m_2-limit point of Ω^c and that z is a limit point of the set of nondegenerate components of $\partial\Omega$; then $C^\infty(\overline{\Omega})$ is not dense in $W^{k,p}(\Omega)$, for any $k \geq 1$ and any $1 \leq p < \infty$.*

Remark. The non degeneracy assumption on the boundary components is needed, at least when $p > 2$. See Example 7.5.

Proof. Since $W^{k,p}$–norms increase with k, it suffices to establish the theorem for $k = 1$. Fix a p with $1 \leq p < \infty$. Let z be a limit point of the nondegenerate components of $\partial\Omega$ and $t > 0$, and assume that $m_2(B(z,t) \setminus \Omega) = 0$. Then there exists $w \in B(z,t/2)$ belonging to a nondegenerate component of $\partial\Omega$ such that $m_2(B(w,t/2) \setminus \Omega) = 0$. Applying Proposition 6.1 with K the component of $\overline{B(w,t/4)} \setminus \Omega$ that contains z, we see that there is a disk $B(a,r)$ with the property that $B(a,r/2) \cap \Omega$ has at least two components and $m_2(B(a,r) \setminus \Omega) = 0$. Indeed, this is immediate if case (i) of Proposition 6.1 holds, and if case (ii) holds, then $B(a,r)$ will have the required properties provided a is an interior point of the circular arc and r is sufficiently small.

Let $B(a,r/2) \cap \Omega = U_1 \cup U_2$, where U_1 is a component of the set. By changing coordinates if necessary, we assume that $S_1 = [0,s] \times [0,s] \subset U_1 \cap B(a,r/2)$ and $S_2 = [0,s] \times [\widehat{s}, s+\widehat{s}] \subset U_2 \cap B(a,r/2)$, where $s, \widehat{s} > 0$ and $s + \widehat{s} \leq 1$. Also, put $R = [0,s] \times [0, s+\widehat{s}]$ and observe that $m_2(R \setminus \Omega) = 0$. Let $\varphi \in C^\infty(\mathbb{R}^2)$ be identically 1 on $B(a,r/2)$ and zero on $\mathbb{R}^2 \setminus B(a,r)$, and set $u = \varphi \cdot \chi_{U_1}$. Then $u \in W^{1,p}(\Omega)$, and we will show that u cannot be approximated to arbitrary accuracy in this space by functions in $C^\infty(\mathbb{R}^2)$.

To this end, suppose that $v_n \in C^\infty(\mathbb{R}^2)$ and $\lim \|u - v_n\|_{W^{1,p}(\Omega)} = 0$. Since u is identically 1 on S_1 and 0 on S_2 it follows that

$$\lim_{n\to\infty} \Big[\frac{1}{s^2} \iint_{S_1} |v_n| \, dxdy - \frac{1}{s^2} \iint_{S_2} |v_n| \, dxdy\Big] = 1\,.$$

On the otherhand, since $|\nabla u| = 0$ on $R \cap \Omega$ and $m_2(R \cap \Omega) = m_2(R)$ we have that

$$\begin{aligned}
\frac{1}{s^2} \iint_{S_1} |v_n| \, dxdy - \frac{1}{s^2} \iint_{S_2} |v_n| \, dxdy &\leq \frac{1}{s^2} \iint_{S_1} |v_n(x,y) - v_n(x, y+\widehat{s})| \, dxdy \\
&\leq \frac{1}{s^2} \iint_{S_1} \int_0^{\widehat{s}} |\nabla v_n(x, y+\tau)| \, d\tau dxdy \\
&\leq \frac{1}{s} \int_0^s \int_0^{s+\widehat{s}} |\nabla v_n| \, dydx \\
&= \frac{1}{s} \iint_{R\cap\Omega} |\nabla (v_n - u)| \, dxdy.
\end{aligned}$$

By applying Hölder's inequality, we get that the last term above is dominated by $s^{-1/p} \|u - v_n\|_{W^{1,p}(\Omega)}$, which tends to zero as n tends to infinity. This contradiction proves that $C^\infty(\overline{\Omega})$ is not dense in $W^{1,p}(\Omega)$. This completes the proof of Theorem C.

The next proposition provides another useful necessary condition for approximation of Sobolev functions. The geometric condition on the domain is simpler than that in the Theorem C, however it is necessary to also assume the existence of a certain Sobolev function on the domain. The proposition is closely related to an example of Amick [3]. It is an easy consequence of the Sobolev embedding theorem (see Theorem 5.4, Part II in [1]) and the argument used to prove Lemma 2 in [3]. We omit the proof. This proposition will be used in the next section to construct examples.

Proposition 6.2. *Let Ω be a domain in $\mathbb{R}^2$ containing congruent triangles $T_1, T_2 \subset \Omega$ with disjoint interiors and a common vertex. If $p > 2$ and there exists $u \in W^{1,p}(\Omega)$ such that $u|_{T_1} \equiv 1$ and $u|_{T_2} \equiv -1$, then $C(\mathbb{R}^2) \cap W^{1,p}(\Omega)$ is not dense in $W^{1,p}(\Omega)$.*

7. Examples

We begin this section by constructing, for each $p > 2$, a domain $\Omega \subset \mathbb{R}^2$ for which the functions in $C^\infty(\Omega)$ with bounded gradients are not dense in $W^{1,p}(\Omega)$. Of course, by Theorem A, Ω will not satisfy the interior segment condition. We then use Ω to produce a domain $\tilde{\Omega} \subset \mathbb{R}^3$ which does satisfy the interior segment condition and for which all points in Ω^c are m_3 limit points of Ω^c, and yet functions in $C^\infty(\tilde{\Omega})$ with bounded gradients are not dense in $W^{1,p}(\tilde{\Omega})$. Thus any extension of Theorem A or Theorem B to $\mathbb{R}^n$ will require further geometric hypotheses on the domain.

Example 7.1. Fix $p > 2$ and fix a sequence $\{y_j\}$ with $y_0 = 1$, $0 < y_{j+1} < y_j$ for $j \geq 0$, and $\lim y_j = 0$. Let $\{b_j\}$ be a sequence of positive numbers satisfying

$$b_j < y_{j-1} - y_j\,, \qquad j \geq 1 \tag{7.1}$$

and

$$\sum_{j=1}^{\infty} y_j^{2(1-p)} b_j < \infty\,. \tag{7.2}$$

Define $\Omega \subset \mathbb{R}^2$ to be

$$\Omega = \{(x,y) \in (-2,2) \times (0,2) \mid y < \sqrt{|x|}\} \cup \bigcup_{j=1}^{\infty} \{(x,y) \mid y_j < y < y_j + b_j, |x| < 2\}\,.$$

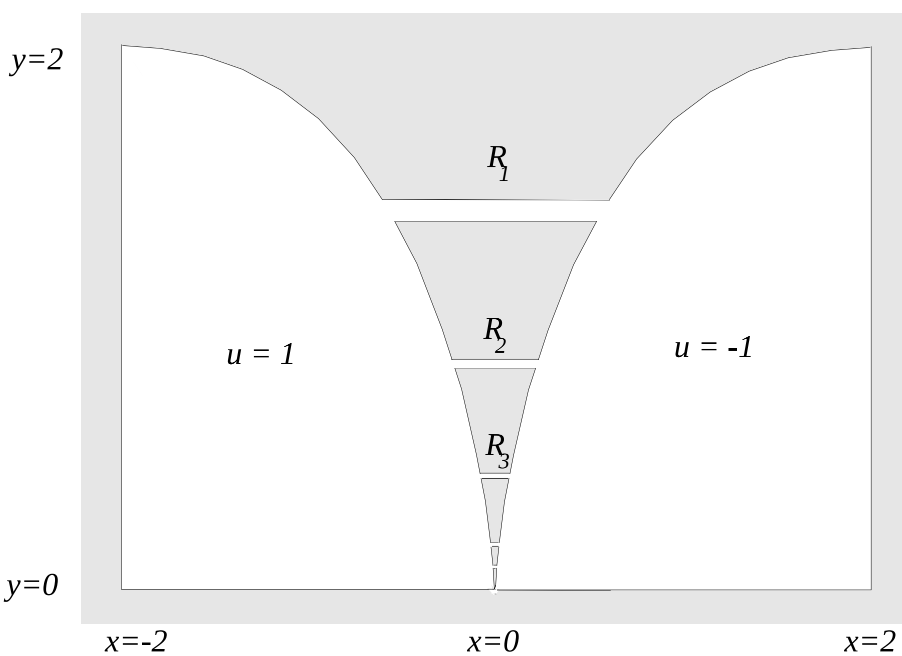

FIGURE 5. $C_b^\infty(\Omega)$ is not dense in $W^{k,p}(\Omega)$ for any $p > 2$.

For $j \geq 1$, set

$$R_j = [-y_j^2, y_j^2] \times [y_j, y_j + b_j]\,,$$

and note that, by (7.1), the rectangles $\{R_j\}$ are disjoint.

Define a function u on Ω by

$$u(x,y) = \begin{cases} 1 & \text{if } x > 0 \text{ and } (x,y) \in \Omega \setminus \cup R_j \\ -1 & \text{if } x < 0 \text{ and } (x,y) \in \Omega \setminus \cup R_j \\ x/y_j^2 & \text{if } (x,y) \in R_j, \quad j \geq 1\,. \end{cases}$$

Clearly, $u \in W^{1,p}(\Omega)$ since $\sum y_j^2 b_j y_j^{-2p} < \infty$ by (7.2).

It is a consequence of Proposition 6.2 that $C(\mathbb{R}^2) \cap W^{1,p}(\Omega)$ is not dense in $W^{1,p}(\Omega)$ for $p > 2$. However, we wish to show moreover that, for this domain, functions in $C^\infty(\Omega)$ with bounded gradients can not be used to approximate u.

Suppose that $v \in C^\infty(\Omega)$ and $|\nabla v|$ is bounded. This implies that, for all sufficiently large j, either $v(y_j^2, y_j + b_j/2) < 1/4$ or $v(-y_j^2, y_j + b_j/2) > -1/4$. So fix a j for which, without loss of generality, $v(y_j^2, y_j + b_j/2) < 1/4$. There is no essential difference in the remaining argument in the case that $v(-y_j^2, y_j + b_j/2) > -1/4$. We introduce polar coordinates with the origin at $(y_j^2, y_j + b_j/2)$ and such that the sector $S = \{[r,\theta] \mid 0 \leq r \leq 1, 0 \leq \theta \leq \pi/4\} \subset \Omega$.

Now, since u is identically 1 on S, by taking ϵ sufficiently small, we can assure that if $\|u - v\|_{L^p(\Omega)} < \epsilon$, then $\sup\{v[r,\theta] \mid 0 \leq r \leq 1\} > 3/4$ for $\theta \in E \subset [0, \pi/4]$, where $|E| > \pi/8$. But then, for $\theta \in E$,

$$\begin{aligned} 1/2 &< \int_0^1 |\nabla v[r,\theta]|\, dr \\ &\leq \Big(\int_0^1 |\nabla v[r,\theta]|^p r\, dr\Big)^{1/p} \Big(\int_0^1 \frac{1}{r^{1/(p-1)}}\, dr\Big)^{(p-1)/p}. \end{aligned}$$

Thus, since we have assumed that $p > 2$, if $\|u - v\|_{L^p(\Omega)} < \epsilon$, then $\int_S |\nabla v|^p$ is bounded below by a positive constant depending only on p. Since ∇u is identically zero on S, this shows that

$$\|u - v\|_{W^{1,p}(\Omega)} \geq C_0 > 0\,, \tag{7.3}$$

if $v \in C^\infty(\Omega)$, $|\nabla v|$ is bounded, and $p > 2$.

Remark. We remark that the argument used to establish (7.3) could also be used to give a prove of Proposition 6.2 which does not use the Sobolev Embedding Theorem. Finally, we point out that it is an immediate consequence of Theorem 8, using δ-modifications at the origin, that $C^\infty(\mathbb{R}^2)$ is dense in $W^{1,p}(\Omega)$ for $1 \leq p \leq 2$.

Using Example 7.1, it is easy to produce domains in $\mathbb{R}^n$, $n \geq 3$, for which Theorem A and Theorem B both fail.

Example 7.2. Now, again for $p > 2$ fixed, let Ω be the domain constructed above, and define $\tilde{\Omega} = (0,1) \times \Omega \subset \mathbb{R}^3$. Then each point of $\tilde{\Omega}^c$ is an m_3-limit point of $\tilde{\Omega}^c$ and $\tilde{\Omega}$ is a basic interior segment domain (as in Section 2). Let $\tilde{u}(x,y,z) = u(y,z)$, $(x,y,z) \in \tilde{\Omega}$, where u is the function defined above. Let $\tilde{v} \in C^\infty(\tilde{\Omega})$ and suppose that $|\nabla \tilde{v}|$ is bounded. Then

$$\|\tilde{u} - \tilde{v}\|^p_{W^{1,p}(\tilde{\Omega})} \geq C \int_0^1 \|u - \tilde{v}(\cdot, t)\|^p_{W^{1,p}(\Omega)}\, dt \geq C C_0^p > 0 \,,$$

by (7.3). Thus $\tilde{u}$ is not the limit in $W^{1,p}(\tilde{\Omega})$ of functions in $C^\infty(\tilde{\Omega})$ with bounded gradients and hence the global C^∞ functions are not dense.

Remark. The construction of the domain in Example 7.2 may be modified to produce a domain that is starshaped instead of satisfying the interior segment condition. Thus there is a starshaped domain $\Omega \in \mathbb{R}^3$ such that each point of Ω^c is an m_3-limit point of Ω^c, and yet $C_b^\infty(\Omega)$ is not dense in $W^{1,p}(\Omega)$ for $p > 2$.

We next give an example of a planar domain that is bounded, simply connected, the complement of the closure is connected, each point of the complement is an m_2-limit point of the complement, and for which $C(\mathbb{R}^2) \cap W^{1,p}(\Omega)$ is not dense in $W^{1,p}(\Omega)$ for $p > 2$.

Example 7.3. Let $E \subset [-1,1]$ be a Cantor set of positive length containing the points -1 and 1; that is, E is a perfect set which is nowhere dense. We further require that every point of E be an m_1-limit point of E. Define Ω to be the domain

$$\Omega = [-2,2] \times [0,2] \setminus \bigcup_{e \in E} s[(0,0),(e,1)] \,,$$

where s[a,b] denotes the closed line segment in $\mathbb{R}^2$ from a to b. It is clear that Ω satisfies all the requirements stated for this example.

Let $u \in C_b^\infty(\Omega)$ satisfy, for $(x,y) \in \Omega$,

$$u(x,y) = \begin{cases} 1 & \text{if } x > y \text{ and } x^2 + y^2 < 1/2 \\ -1 & \text{if } x < -y \text{ and } x^2 + y^2 < 1/2 \\ 0 & \text{if } -y < x < y \text{ or } x^2 + y^2 > 1 \,. \end{cases}$$

Clearly $u \in W^{1,p}(\Omega)$, for all $p < \infty$.

Proposition 6.2 now asserts that $C(\mathbb{R}^2) \cap W^{1,p}(\Omega)$ is not dense in $W^{1,p}(\Omega)$ for $p > 2$. We point out, however, that if $1 \leq p \leq 2$, then $C^\infty(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$. This can be seen by performing a δ-modification on Ω at the origin, and using the function ϕ from Lemma 5.1 as was done in the proof of Theorem 8. A polar coordinate-type mapping takes the resulting domain onto a domain satisfying the assumptions of Theorem 5, and so, as in the proof of Theorem 8, $C^\infty(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$ for $1 \leq p \leq 2$.

Example 7.4. In this example, Ω satisfies the geometric and density conditions of the previous example but has infinitely many two sided boundary points. Moreover, $C^\infty(\overline{\Omega})$ is not dense in $W^{k,p}(\Omega)$ for any p.

It is possible to construct Cantor sets K_n in $(-2,0)$ for each $n = 1, 2, \ldots$ so that the set $E = \cup K_n$ satisfies:

$$0 < |I \cap E| < 1 \qquad \text{for all intervals } I \subset (-2,0) \,.$$

Let $f = \sum 2^{-n}\chi_{K_n}$, so that f is an upper semicontinuous function and

$$\Omega = (-2,2)\times(-1,1) \setminus \bigcup_{x\in(-2,0)} s[(x,-f(x)),(x,f(x))]$$

is an open set satisfying the above conditions. If $u \in W^{k,p}(\Omega)$ and $u(x,y)$ is equal to 1 for $(x,y)\in\Omega$ with $y>0$ and $x<x_0<0$ and $u(x,y)=-1$ for $(x,y)\in\Omega$ with $y<0$ and $x<x_0<0$, then u can not be approximated by a $C^\infty(\overline{\Omega})$ function. (See [8] for a similar example.)

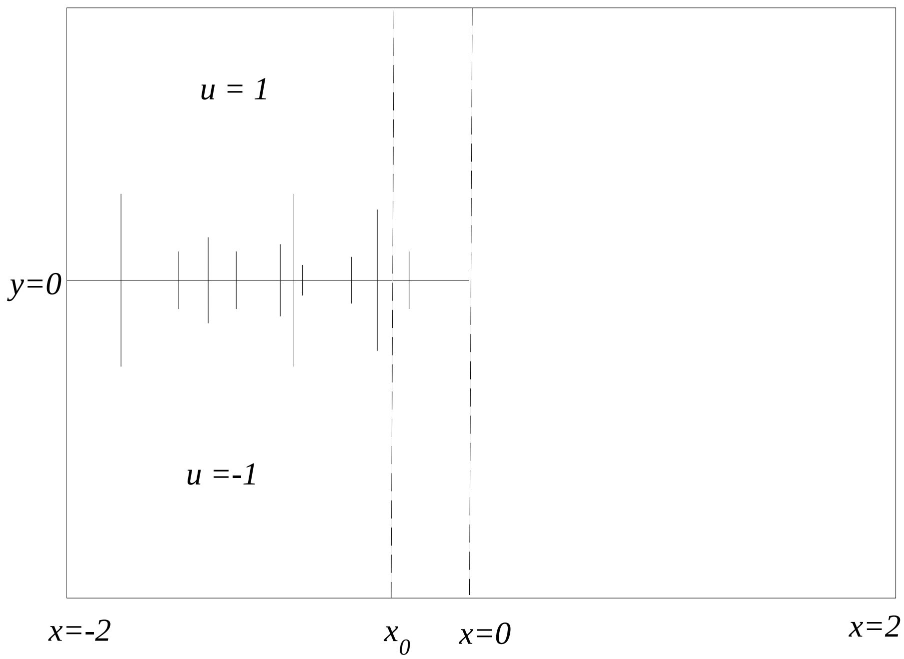

FIGURE 6. $C^\infty(\overline{\Omega})$ is not dense in $W^{k,p}(\Omega)$ for any p.

The domain Ω in the next example also satisfies the geometric condition from Proposition 6.2, but now $C^\infty(\mathbb{R}^2)$ will fail to be dense in $W^{1,p}(\Omega)$ for $1\le p<2$, in contrast to Example 7.3.

Example 7.5. Using a function similar to the argument function it is possible to construct a function f on the open equilateral triangle $T\subset\mathbb{R}^2$ with vertices $(0,0)$, $(1,0)$ and $(1/2,\sqrt{3}/2)$ such that:

(i) $0\le f\le 1$;
(ii) $\int_T |\nabla f|^p\,dxdy = C(p)<\infty$, $1\le p<2$;
(iii) f extends to be smooth on $\overline{T}\setminus\{(0,0),(1,0)\}$, with $f(x,0)=0$, $0<x<1$, and $f(x,y)=1$ for $(x,y)\in\partial T$ and $y>0$.

Fix a p with $1\le p<2$. Let I_0 be the interval $(-2,-1)$ and I_1 the interval $(1,2)$, and let $E\subset[-1,1]$ be a Cantor set with $|E|>0$ and such that the complementary intervals, $[-1,1]\setminus E = \cup_{j=2}^\infty I_j$, satisfy

$$\sum_{j=2}^{\infty} |I_j|^{2-p} < \infty. \tag{7.4}$$

Denote by T_j the open equilateral triangle in the upper-half plane with one edge I_j. Since T_j can be obtained from T by a translation and a dilation, these operations can be used with the function f described above to define a function f_j on T_j such that $0 \le f_j \le 1$, $\int_{T_j} |\nabla f_j|^p \, dx = C(p)|I_j|^{2-p}$ and the natural analog of (iii) holds for f_j on T_j as well.

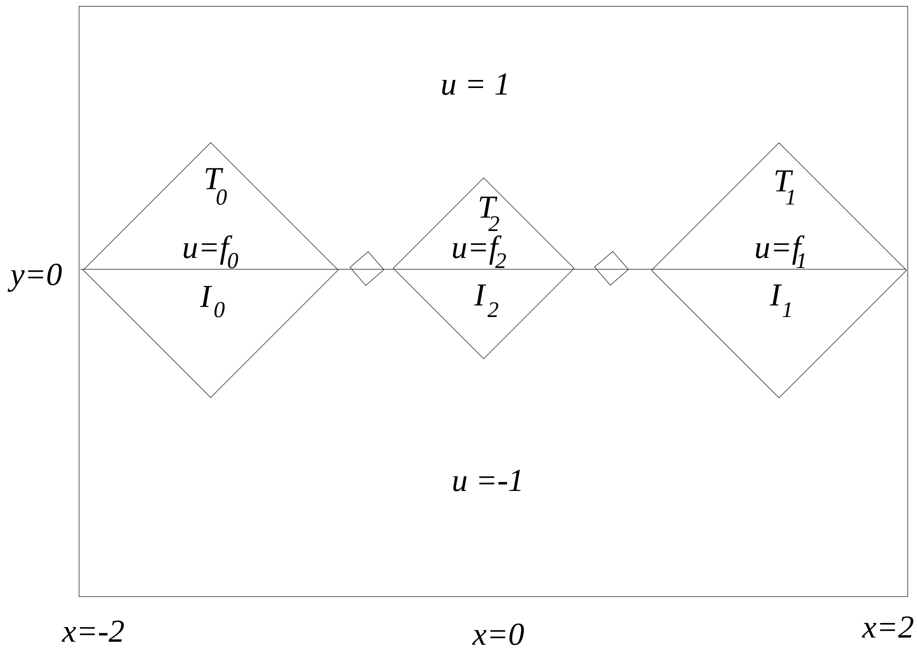

FIGURE 7. $C^\infty(\overline{\Omega})$ is not dense in $W^{k,p}(\Omega)$ for $1 \le p < 2$.

Let $\Omega \subset \mathbb{R}^2$ be the domain $(-2,2) \times (-1,1) \setminus [E \times \{0\}]$. Observe that Theorem C does not apply to Ω since all but one of the boundary components of Ω are degenerate. Nevertheless, $C^\infty(\overline{\Omega})$ fails to be dense in $W^{1,p}(\Omega)$ for $1 \le p < 2$. To see this, let $u \in C(\Omega)$ be such that

$$u(x,y) = \begin{cases} f_j(x,y) & \text{if } (x,y) \in T_j, \quad j \ge 0 \\ 1 & \text{if } (x,y) \in \Omega \setminus \cup_{j=0}^{\infty} T_j \text{ and } y > 0 \\ -u(x,-y) & \text{if } (x,y) \in \Omega \text{ and } y < 0\,. \end{cases}$$

Then

$$\begin{aligned} \int_\Omega |\nabla u|^p \, dxdy &= 2 \sum_{j=0}^{\infty} \int_{T_j} |\nabla f_j|^p \, dxdy \\ &= 2 \sum_{j=0}^{\infty} C(p)|I_j|^{2-p} \\ &< \infty\,, \end{aligned}$$

by (7.4). Thus, using the absolute continuity of f on horizontal and vertical lines from (iii), by Theorem 1.1.3.2 in [10] and the inequality $|u| \le 1$, we have that $u \in W^{1,p}(\Omega)$.

Let $v \in C^\infty(\overline{\Omega})$. An easy argument shows that if $\int_\Omega |u-v|^p \, dxdy$ is sufficiently small, then

$$\int_{-1}^{1} |\nabla v(x,y)| \, dy \geq 1, \qquad x \in F,$$

where $F \subset E$ satisfies $|F| \geq |E|/2$. Thus by Hölder's inequality,

$$\begin{aligned}\int_{(E\times[-1,1])\cap\Omega} |\nabla(u-v)|^p \, dxdy &= \int_{E\times[-1,1]} |\nabla v|^p \, dxdy \\ &\geq \frac{|E|}{2}\frac{1}{2^{p-1}},\end{aligned}$$

if $\int_\Omega |u-v|^p \, dxdy$ is sufficiently small. Hence $C^\infty(\overline{\Omega})$ is not dense in $W^{1,p}(\Omega)$ for $1 \leq p < 2$.

In case $p > 2$, we can appeal to the Sobolev embedding theorem to show that $C^\infty(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$. Indeed, by the Sobolev embedding theorem (see Theorem 5.4, Part II on page 98 in [1]), the restriction of any function $u \in W^{1,p}(\Omega) \cap C^\infty(\Omega)$ to either of the rectangles $\Omega \cap \{y > 0\}$ or $\Omega \cap \{(x,y) \mid y < 0\}$ must extend to be a continuous function on the corresponding closure. Since $\Omega \cap \{y = 0\}$ is dense in $[-2,2] \times \{0\}$, it follows that u extends to be uniformly continuous on the rectangle $\overline{\Omega}$. To see that this extension of u is in $W^{1,p}((-2,2)\times(-1,1))$, we again appeal to Section 1.1.3 in [10]. The original function u must have been absolutely continuous on almost every vertical line inside $\Omega \cap \{y > 0\}$ or $\Omega \cap \{y < 0\}$, and so the continuity of the extended function shows that the same is true for this function on almost every vertical line inside $(-2,2)\times(-1,1)$. Hence the extension of u is in $W^{1,p}((-2,2)\times(-1,1))$, and the desired density result now follows from the corresponding property for rectangles which, for example, trivially satisfy the segment condition.

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