WEAK SLICE CONDITIONS AND HÖLDER IMBEDDINGS

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Abstract. We introduce weak slice conditions and investigate imbeddings of Sobolev spaces in various Lipschitz-type spaces.

0. Introduction

Bojarski [B] proved that Sobolev-Poincaré imbeddings are valid on all John domains; see also [Mto]. In [BK1], it is shown that John domains are essentially the right class for this imbedding, since a bounded domain \( G \subset \mathbb{R}^n \) is a John domain if and only if it supports a Sobolev-Poincaré imbedding and satisfies a certain separation condition. Corresponding results for the \( p = n \) (Trudinger) and \( p > n \) (Hölder) cases of the Sobolev Imbedding Theorem are given in [BK2], where it is shown that for domains satisfying a certain slice condition, each of these imbeddings is equivalent to a mean cigar condition dependent on \( p \); see Section 1 for definitions of these concepts. For other results on Hölder imbeddings, we refer the reader to [A], [Mz], and [KR].

In one way, the results of [BK2] are less satisfying than those of [BK1]. In [BK1], the strong geometric condition (John) is equivalent to the combination of the weak geometric condition (separation) and a Sobolev-Poincaré imbedding. However in [BK2], the weak geometric condition (slice) is not implied by the strong geometric condition (mean cigar) for any value of \( p \geq n \). For \( p = n \), Buckley and O'Shea [BO] overcame this deficiency by showing that the strong geometric condition is equivalent to the combination of a so-called weak slice condition (which is implied by a slice condition) and the Trudinger imbedding. Here we prove the following analogous result for \( p > n \); the terminology is explained in Sections 1 and 2.

Theorem 0.1. Let \( 0 < \alpha < 1 \) and \( G \subset \mathbb{R}^n \). Then \( G \) is an (inner) \( \alpha \)-mCigar domain if and only if it is an (inner) \( \alpha \)-wSlice domain which supports an (inner) \( p \)-Hölder imbedding for \( p = (n - \alpha)/(1 - \alpha) > n \).

More generally, we prove variations of Theorem 0.1 with the Euclidean metric replaced by some other metric (for instance the inner Euclidean metric), and we

2000 Mathematics Subject Classification. 46E35, 30C65.

This paper was begun when both authors were visiting the University of Jyväskylä in the summer of 1998. They both wish to thank the Mathematics Department for its hospitality and support. The first author was partially supported by Enterprise Ireland.

\(^{1}\)A statement with one or more parenthesized instances of “inner” is meant to be expanded into two statements: one includes “inner” in all locations, and the other excludes it everywhere.

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investigate related imbeddings of $W^{1,p}(G)$ in Lipschitz spaces with respect to one of a large class of metrics (and even some non-metrics), generalizing results of [KR]. Reflecting the fact that the strong geometric condition is genuinely different for each $p \geq n$, our weak slice conditions (all weaker than the slice condition of [BK2]) will also be genuinely different for each value of $\alpha$.

The rest of the paper is organized as follows. After some preliminaries, we introduce the slice conditions in Section 2, where we also prove some basic related results. The imbedding theorems are stated and proved in Section 3, and finally we look at some specific examples in Section 4.

We note that the weak slice theory developed here is used in [BS] to investigate what product domains are quasiconformal images of balls or other nice domains.

1. Preliminaries

1.1. Notation.

We adopt two common conventions. First, we drop parameters if we do not wish to specify their values; for instance, we define $C$-uniform domains, but often talk about uniform domains. Second, we write $C = C(x, y, \ldots)$ to mean that a constant $C$ depends only on the parameters $x, y, \ldots$.

If $S \subset \mathbb{R}^n$ is measurable, then $|S|$ is the Lebesgue measure of $S$, and $u_S$ is the average value of a function $u$ on $S$. $H^k$ denotes $k$-dimensional Hausdorff measure. In proofs, we write $A \lesssim B$ if $A \leq CB$ for some constant $C$ dependent only on allowed parameters; we write $A \approx B$ if $A \lesssim B \lesssim A$. We write $A \wedge B$ and $A \vee B$ for the minimum and maximum, respectively, of the quantities $A$ and $B$. Unless otherwise stated, $G$ is a proper subdomain of $\mathbb{R}^n$.

For this paragraph and the next, $U \subset \mathbb{R}^n$. Given $x, y \in U$, we define $\delta_U(x)$ to be the distance from $x$ to $\partial U$, and $\Gamma_U(x, y)$ to be the class of rectifiable paths $\lambda : [0, t] \to U$ for which $\lambda(0) = x$ and $\lambda(t) = y$. If $\gamma$ is a rectifiable path in $U$, and $\alpha \in \mathbb{R}$, and $ds$ is arclength measure, we define

$$\text{len}_{\alpha, U}(\gamma) = \int_{\gamma} \delta_U^{-\alpha}(z) ds(z),$$

together with an associated metric

$$d_{\alpha, U}(x, y) = \inf_{\gamma \in \Gamma_U(x, y)} \text{len}_{\alpha, U}(\gamma), \quad x, y \in U.$$

Of course, $d_{\alpha, U}(x, y) = \infty$ unless $x, y$ lie in the same path component of $U$. We write $\text{len}$ in place of $\text{len}_{1, U}$; note that $d_{1, U}$ is the inner Euclidean metric. For the sake of brevity, it is convenient to abuse notation by, for instance, writing $\text{len}_{\alpha, U}(\gamma \cap S)$ for the $d_{\alpha, U}$-length of those parts of a path $\gamma$ lying in a subset $S$ of $U$. We write $[x, y]$ for the line segment joining a pair of points in $\mathbb{R}^n$, $[x \to y]$ for the path parametrized by arclength that goes from $x$ to $y$ along $[x, y]$.

We are mainly interested in $d_{\alpha, U}$ when $\alpha \in [0, 1]$ and $U$ is a domain. When $U$ is a domain and $\alpha \leq 0$, $d_{\alpha, U}$-geodesics exist for every pair of points; see [GO], [Mtn]. However, they can fail to exist for any choice of $\alpha \in (0, 1]$. We cannot find a reference for this fact, so we now pause to give a counterexample (the case $\alpha = 1$ is of course trivial).
Example 1.2. The desired domain will consist of the unit disk $B = B(0, 1)$ with certain segments of the real axis being removed. Let $z = (0, 1/2)$ and $G_0 \equiv B \setminus \{(-t, 0), (t, 0)\}$, where $0 < t < 1$ is so close to 1 that $d \equiv d_{\alpha, G_0}(z, -z)$ is strictly larger than the value $d_*$ of the (convergent improper integral which defines the) $d_{\alpha, G_0}$-length of the linear segment $[z, -z]$. Next, let $(x_k)_{k=1}^{\infty}$ be a strictly decreasing sequence with limit zero and $x_1 < t$. We write $z_k = (x_k, 0) \in B$, $h_1 = t - x_1$, and $h_k = x_{k-1} - x_k$, $k > 1$. Let $d_{0,j}$ denote the infimum of the (convergent improper integrals which define the) $d_{\alpha, G_0}$-lengths of paths from $z$ to $-z$ that pass through $z_j$, $j \in \mathbb{N}$. By taking $x_1$ to be small enough, we may assume that $d_{0,1} < d$.

We define $I_k = (z_k - (\epsilon_k, 0), z_k + (\epsilon_k, 0))$, $k \in \mathbb{N}$, and $G_m = G_0 \cup \bigcup_{k=1}^{m} I_k$ for each $k \in \mathbb{N} \cup \{\infty\}$; the desired domain $G$ will be $G_{\infty}$. Here $\epsilon_k$ is positive, less than $(h_k \wedge h_{k+1})/2$ (so that the intervals $I_k$ are disjoint), and so small that

$$d > d_{k,1} > d_{k,2} > \cdots > d_{k,j} > \cdots > d_*, \quad k \in \mathbb{N}$$  \hspace{1cm} (1.3)

and

$$(1 - (k + 1)^{-2}) \leq \frac{d_{k,j} - d_*}{d_{k-1,j} - d_*} \leq 1, \quad k \in \mathbb{N}$$  \hspace{1cm} (1.4)

where for each $j \in \mathbb{N}$, $k \in \mathbb{N} \cup \{\infty\}$, $d_{k,j}$ is the infimum of the $d_{\alpha, G_k}$-lengths of paths from $z$ to $-z$ that pass through $I_j$ (if $j \leq k$) or through $z_j$ (if $j > k$). We leave to the reader the task of verifying that these inequalities are satisfied for appropriate $\epsilon_k > 0$ and that (1.4) implies that $\lim_{k \to \infty} d_{k,j} = d_{\infty,j} > d_*$. It is also not hard to see that $d_{\alpha, G_{\infty}}(z, -z) = d_*$, but that there is no geodesic.

Given $x \in U$, $E, F \subset U$, and a metric $\rho$ on $U$, we write $d_\rho(E, F)$ for the $\rho$-distance between $E$ and $F$, $\text{dia}_\rho(E)$ for the $\rho$-diameter of $E$, and $B_\rho(x, r) = \{y \in U : d_\rho(x, y) < r\}$. If $\rho = d_1, U$, we instead write $d_U(E, F)$, $\text{dia}_U(E)$, and $B_U(x, r)$ for these concepts, while if $\rho$ is the Euclidean metric (and so $U = \mathbb{R}^n$), we write $d(E, F)$, $\text{dia}(E)$, and $B(x, r)$. For brevity, we define $d_U = d_{1, U}$; in particular, $d_{\mathbb{R}^n}$ is the Euclidean metric. Note that distance to the boundary of $U$ is the same with respect to $d_{\mathbb{R}^n}$ and $d_U$, and that $B_U(x, r) = B(x, r)$ if $r \leq \text{dia}(U)$.

1.5. Function spaces and Hölder-type imbeddings.

For $n < p < \infty$, $L^{1,p}(G)$ is the space of functions $f : G \to \mathbb{R}$ with distributional gradients in $L^p(G)$, and $W^{1,p}(G) = L^p(G) \cap L^{1,p}(G)$ is the corresponding Sobolev space. We write $\|u\|_{L^{1,p}(G)} = \|\nabla u\|_{L^p(G)}$ and $\|u\|_{W^{1,p}(G)} = \|u\|_{L^p(G)} + \|u\|_{L^{1,p}(G)}$.

Let $0 < t \leq 1$, $0 < \epsilon \leq \infty$, and let $d : G \times G \to [0, \infty)$ be a function which is positive off the diagonal. We define $\text{Lip}_{t, \epsilon}(G, d)$ to be the space of all functions $u : G \to \mathbb{R}$ for which

$$\|u\|_{\text{Lip}_{t, \epsilon}(G, d)} \equiv \sup_{0 < |x - y| \leq r} \frac{|u(x) - u(y)|}{d(x, y)^t} < \infty.$$

If $d(x, y) = |x - y|$, we simply write $\text{Lip}_{t, \epsilon}(G)$; note that $\text{Lip}_{t, \epsilon}(G, d)$ is the familiar space of functions which are Hölder continuous of order $t$. If $\epsilon \geq \text{dia}(G)$, we omit the “$\epsilon$” subscript. We also define $C^{0,t}(G, d)$ to be the space of all bounded functions in $\text{Lip}_{t}(G, d)$ and write

$$\|u\|_{C^{0,t}(G, d)} = \|u\|_{L^\infty(G)} + \|u\|_{\text{Lip}_{t}(G, d)}.$$
The notation $C^{0,d}(G, d)$ is not very appropriate unless we at least have $d(x, y) \to 0$ as $x \to y$, but we do not wish to put restrictions on $d$ at this point.

For any pair of these spaces, we write $A \leftrightarrow B$ if $A \subset B$ and $\| \cdot \|_B \leq C \| \cdot \|_A$. We call the smallest constant $C$ for which this condition is valid the imbedding constant for $A \leftrightarrow B$. Of course this quantity and the various quantities $\| \cdot \|_A$ are not in general “norms” in the functional analysis sense, but the notation is still convenient. We are particularly interested in the imbedding $L^{1,p}(G) \hookrightarrow \text{Lip}_{1-n/p}(G, d)$, and say that $G$ supports the $(p, C)$-Hölder imbedding if this imbedding holds with imbedding constant $C$. In particular, we say that $G$ supports the $(p, C)$-Hölder imbedding (or inner $(p, C)$-Hölder imbedding) if $d = d_{\mathbb{R}^n}$ (or $d = d_G$, respectively).

It is well-known that for $p > n$, balls and other “nice” domains support $(p, C)$-Hölder imbeddings, with $C = C(n, p)$. More generally, it is shown in [BK2] that this imbedding also holds on all $\alpha$-mCigar domains (as defined below) where $\alpha = (p - n)/(p - 1)$; it is also shown there that the imbedding implies the $\alpha$-mCigar condition if the domain satisfies a (strong) slice condition, as defined below.

### 1.6. Uniform domains and mean cigar domains.

Let $C \geq 1$ and let $d$ be the Euclidean metric. We say that a domain $G$ is a C-uniform domain if for every pair $x, y \in G$, there is a C-uniform path, i.e., a path $\gamma \in \Gamma_G(x, y)$ of length $l$ and parametrized by arclength for which $l \leq C d(x, y)$, and $t \wedge (l - t) \leq C \delta_G(\gamma(t))$. An inner C-uniform domain is defined similarly but with $d = d_G$. All uniform domains are inner uniform, while a slit disk is inner uniform but not uniform. For more on inner uniform domains, see [V].

Suppose that $0 \leq \alpha \leq 1 \leq C$ and let $d : G \times G \to [0, \infty)$. We say that $G$ is an $(\alpha, C; d)$-mCigar domain if for every pair $x, y \in G$, there is a $(\alpha, C; d)$-mCigar path, i.e., a path $\gamma \in \Gamma_G(x, y)$ such that

\[
\begin{align*}
\text{len}_{\alpha, G}(\gamma) & \leq C d(x, y)^{\alpha} & 0 < \alpha \leq 1, \\
\text{len}_{0, G}(\gamma) & \leq C \log[1 + d(x, y)/(\delta_G(x) \wedge \delta_G(y))], & \alpha = 0.
\end{align*}
\]

In particular, if $d$ is the Euclidean metric, we simply say that $G$ is an $(\alpha, C)$-mCigar domain, while if $d = d_G$, we say that $G$ is an inner $(\alpha, C)$-mCigar domain. In practice we shall not use this terminology for $\alpha = 1$: we prefer to use the more common term $C$-quasiconvex domain rather than $(1, C)$-mCigar domain.

All uniform domains are $\alpha$-mCigar domains for any choice of $\alpha \in [0, 1]$. Gehring and Osgood [GO] showed that the classes of $0$-mCigar domains and uniform domains coincide, and Väisälä [V, 2.33] showed that the classes of inner $0$-mCigar and inner uniform domains coincide. Note that the classes of inner uniform and inner $\alpha$-mCigar domains contain their Euclidean analogues (strictly, since a planar slit disk is in all of the inner classes but none of the Euclidean classes).

The role of the parameter $\alpha$ in the definition of an $\alpha$-mCigar domain is rather subtle. First we note that the class of (inner) $\alpha'$-mCigar domains includes the class of (inner) $\alpha$-mCigar domains if and only if $\alpha \leq \alpha'$. For the Euclidean case, see [L] and [BK2]; the inner case is similar. Lappalainen’s examples [L, 6.7] of (non-uniform) domains that are $\beta$-mCigar but not $\alpha$-mCigar, for each possible choice of $0 < \alpha < \beta \leq 1$ makes use of a rather elaborate Cantor-type construction. It
is intuitively clear that any such example must have a similar level of complexity. Thus domains that are easy to describe explicitly typically are either in all these classes or none of them. Among the examples of domains which are 0-mCigar, and so in all $\alpha$-mCigar classes, are all bounded Lipschitz domains, as well as some domains with fractal boundary, such as the interior of a von Koch snowflake. In Section 4 we will give some further examples of mCigar domains.

We refer the reader to [BK2], [GM], and [L] for more information about $\alpha$-mCigar domains, which elsewhere go under the aliases “weak cigar domains” and “Lip$\alpha$ extension domains”. We use the term “mean cigar” because these conditions imply the existence of a path $\gamma$ that satisfies a type of cigar condition on average; see [BK2, Lemma 2.2] and also [BS, Lemma 4.3]. In this paper, we reserve the adjective “weak” for the slice conditions defined in the next section which, in particular, are satisfied by all planar simply-connected domains. By contrast mean cigar conditions rather strongly restrict the geometry of the domain: for instance, the proof of Proposition 4.6 will show that mCigar domains possess neither internal nor external cusps.

2. Slice domains

The conditions defined in Section 1 rather strongly restrict the geometry. For instance, among planar domains, inner uniform domains cannot have external cusps, while uniform and mCigar domains can have neither internal nor external cusps. By contrast, the slice conditions that we define in this section are all quite weak, at least in two dimensions: they are satisfied by any domain quasiconformally equivalent to a uniform domain and hence by all simply-connected planar domains.

2.1. Weak slice domains.

The basic Euclidean 0-wSlice condition defined below is essentially taken from [BO], where it is assumed uniformly for all $x$ and a fixed $y$, but the $\alpha > 0$ case and non-Euclidean variants have not been considered before. We also prove some basic properties of these weak slice conditions in this subsection. The adjective “weak” refers to the fact that for all $\alpha$, an $\alpha$-wSlice condition is implied by the analogous “strong” slice condition which we define later.

Suppose $0 \leq \alpha < 1 \leq C$ and let $d$ be a metric on $G$ satisfying $d_{\mathbb{R}^n} \leq d \leq d_G$. Then $G$ is an $(\alpha, C; d)$-wSlice domain if every pair $x, y \in G$ satisfies the following $(\alpha, C; d)$-wSlice condition: there exist a path $\gamma \in \Gamma_G(x, y)$, pairwise disjoint open subsets $\{S_i\}_{i=1}^m$ of $G$, $m \geq 0$, and numbers $d_i \in [\text{dia}_{d}(S_i), \infty)$ such that:

$$\text{len}(\lambda \cap S_i) \geq d_i / C, \quad \lambda \in \Gamma_G(x, y), 1 \leq i \leq m; \quad \text{(WS-1)}$$

$$\text{len}_{\alpha, G}(\gamma) \leq C(\delta^\alpha_G(x) + \delta^\alpha_G(y) + \sum_{i=1}^m d_i^\alpha); \quad \text{(WS-2)}$$

$$(B(x, \delta_G(x)/C) \cup B(y, \delta_G(y)/C)) \cap S_i = \emptyset, \quad 1 \leq i \leq m. \quad \text{(WS-3)}$$

If $d$ is the Euclidean metric, we say that $G$ is an $(\alpha, C)$-wSlice domain, while if $d = d_G$, we say that $G$ is an inner $(\alpha, C)$-wSlice domain.
We will soon be looking carefully at some of the consequences of this condition and the inter-relationships between its three parts. We point out first of all that by (WS-1), each of the weak slices $S_i$ must separate $x$ from $y$ in the domain $D$. The $\alpha$-wSlice conditions are all (strictly) weaker than the slice conditions which were introduced by the first author and Koskela [BK2], as is shown in Lemma 2.8 below. This latter class is already quite vast since it includes (by Theorem 3.2 in [BK2]) all quasiconformal images of uniform domains and in particular, by the Riemann mapping theorem, all simply connected planar domains. In Theorem 3.1 of [BS], we go further to show that "uniform" can be replaced by "inner uniform" in the above result. There is, however, one significant difference between Slice and $\alpha$-wSlice conditions: we shall see that every $\alpha$-mCigar condition for a pair of points implies an $\alpha$-wSlice condition for that pair, but we shall also see in the final section that, for every $\alpha > 0$, there exist non-Slice domains which are nevertheless $\alpha$-mCigar (and so $\alpha$-wSlice) domains.

Before going on, we now present two examples of planar domains $D$ which are not $\alpha$-wSlice. Let us fix $0 < \alpha < 1$ and write $p = (n - \alpha)/(1 - \alpha) > n$. In general, it is a difficult task to show directly from the definition that a domain is not $\alpha$-wSlice, so we shall use Theorem 0.1. Let $B = B(0, 1)$ be the unit disk. For each positive integer $n$, consider the annulus

$$A_n = \{ z \in B : 1/(n + 1) < 1 - |z| < 1/n \}.$$

Inside $A_n$, we delete a finite set of points $P_n$ so that $\delta_{B \setminus P_n}(z) < 2^{-n}$ for each $z$ in $A_n$. Put $D = B \setminus \bigcup_{n=1}^{\infty} P_n$. Since isolated points are removable singularities of $p$-Hölder continuous functions and, for any $\alpha \in (0, 1)$, $B$ supports a $p$-Hölder imbedding, so must the domain $D$. If $D$ were to be an $\alpha$-wSlice domain then Theorem 0.1 would imply that $D$ is also an $\alpha$-mCigar domain. But this is obviously not the case since for example, $\lim_{t \to 1^-} d_{\alpha,D}((0, 0), (t, t)) = \infty$.

A similar example, which the reader may consider "less trivial", is produced by replacing every point $x_{n,i} \in P_n$ by a closed ball $B_{n,i}$ so small that the concentric double dilates of these balls are pairwise disjoint; we again call the resulting domain $D$. Since uniform domains are $W^{1,p}$-extension domains [J, Theorem 1], we can take a function $f \in W^{1,p}(D)$, extend the functions $f|_{2B_{n,i}\setminus \overline{B_{n,i}}}$ to $B_{n,i}$, and glue together these extensions to get an extension $F$ of $f$ with $\|F\|_{W^{1,p}(B)}$ comparable to $\|f\|_{W^{1,p}(D)}$. It follows from classical results that $F \in \text{Lip}_{1-n/p}(B)$, and so $f \in \text{Lip}_{1-n/p}(D)$. This argument together with Theorem 3.8 implies that $D$ supports a $p$-Hölder imbedding. As before, however, $D$ is not an $\alpha$-mCigar, and hence not an $\alpha$-wSlice, domain.

In the two examples just presented, the obstacles (i.e., removed points or balls) ensure that there are alternative $d_{\alpha,D}$-quasi-optimal paths between pairs of points which are not close in the Hausdorff (set) metric defined induced by the metric $d_{\alpha,D}$. This is the typical situation in which slice conditions fail. Another geometric configuration with this property is a "flat plate" in three (or more) dimensions, i.e. a box which is large in at least two dimensions, but small in at least one other dimension. Showing that a suitably constructed domain with many flat plates is not an $\alpha$-wSlicedomain is, however, rather tricky; we refer the interested reader to the example after Open Problem B in [BS, Section 6].
For $\alpha = 0$, (WS-2) simply says that $\text{len}_{0,G}(\gamma) \leq C(2 + m)$ and so if $\alpha = 0$, we can take $d_i = \text{dia}_d(S_i)$ in the $(\alpha; d)$-wSlice condition. Although not obvious, we shall see below that this can also be done for $\alpha > 0$ (modulo a change in the value of $C$); however allowing inequality is sometimes convenient. For $\alpha = 0$, (WS-3) is an essential part of the definition (lest every domain be a $(0; d)$-wSlice domain), but when $\alpha > 0$ it can be dropped; see [BS, Theorem 4.12]. Obviously an inner $\alpha$-wSlice condition implies an $\alpha$-wSlice condition; the converse is false [BS, Section 5].

Modulo a change in the value of $C$ by a factor at most 4, we may add the following condition to the definition of an $(\alpha, C; d)$-wSlice condition for $x, y$:

$$\text{len}_{\alpha,G}(\gamma \cap S_i) \leq C d_i^\alpha, \quad 1 \leq i \leq m.$$  \hspace{1cm} (WS-4)

To see this suppose that $x, y \in G$ satisfy an $(\alpha, C/4; d)$-wSlice condition for some $C \geq 4$, with slice data $\gamma$, $\{S_i', d_i'\}_{i=1}^m$. We define new $(\alpha, C/2; d)$ slice data $\gamma$, $\{S_i, d_i\}_{i=1}^m$ satisfying (WS-4) as follows. First, we may assume that $2\text{len}_{\alpha,G}(\gamma) \leq C \sum_{i=1}^m (d_i')^\alpha$, since otherwise we simply take $m = 0$. We discard $S_i'$ if $\text{len}_{\alpha,G}(\gamma \cap S_i') > C(d_i')^\alpha$, relabel the remaining ones as $\{S_i\}_{i=1}^m$, and relabel the numbers $d_i'$ in the same fashion, so that $d_i = d_j'$ whenever $S_i = S_j'$. An easy calculation shows that $\sum_{i=1}^m d_i^\alpha \geq \sum_{i=1}^m (d_i')^\alpha/2$, and so we are done.

We now prove a few lemmas concerning weak slice conditions.

**Lemma 2.2.** Suppose that $c \in (0, 1)$, that $x, y$ are points in a bounded domain $G \subset \mathbb{R}^n$, and that $S \subset G \setminus \overline{B_x} \cup \overline{B_y}$ is open, where $B_w \equiv B(w, c\delta_G(w))$, $w \in G$. Suppose further that every $\gamma \in \Gamma_G(x, y)$ intersects $S$. Then $\text{dia}(S) > 2c\delta_G(w)/(1 + c)$ for every $w \in S$.

**Proof.** Suppose for the purpose of contradiction that the lemma is false. Choose $z \in S$ such that $\delta = \delta_G(z) = \max_{w \in \overline{S}} \delta_G(w)$. Thus $\text{dia}(S) \leq c'\delta$ and $S \subset B(z, c'\delta)$, where $c' = 2c/(1 + c) \in (0, 1)$. Let us get a contradiction first under the additional assumption that $\overline{B_x} \subset B(z, c'\delta)$, which of course implies that $\overline{B_x} \subset B(z, r)$ for some $r < c'\delta$. Since $d(B(x, r), \partial G) \leq d(B_x, \partial G)$, it follows that

$$(1 - c')\delta < \delta - r \leq (1 - c)\delta_G(x),$$

and so $\delta < (1 + c)\delta_G(x)$. But $z \notin B_x$ and so $c'\delta > r \geq 2c\delta_G(x)$, which in turn implies that $\delta \geq (1 + c)\delta_G(x)$, giving the desired contradiction.

In view of the above argument, and a similar one with $y$ replacing $x$, we may assume without loss of generality that there exist points $x' \in \overline{B_x} \setminus B(z, c'\delta)$ and $y' \in \overline{B_y} \setminus B(z, c'\delta)$. Let $\lambda \in \Gamma_G(x, y)$ be a path that has $[x \rightarrow x']$ as an initial segment and a reparametrized $[y' \rightarrow y]$ as a final segment. If $\lambda$ intersects $S$, it must do so on the remaining middle segment, and it must pass through points $x''$, $y''$ of first and last contact with $\overline{B(z, c'\delta)}$. We replace the part of $\lambda$ between $x''$ and $y''$ by a suitably parametrized arc on $\partial B(z, c'\delta)$ between $x''$ and $y''$ to get a path $\gamma$ that avoids $B(z, c'\delta) \cap S$. This contradicts the hypotheses, so the lemma must be true. \hfill \Box

According to the next lemma, (WS-1) and (WS-2) together imply that the slices for a pair of points $x, y \in G$ are “neither too large nor too thin” and cover at least some fixed fraction of the $d_{\alpha,G}$-length of any efficient path from $x$ to $y$. 


Lemma 2.3. If the data $\gamma, \{S_i, d_i\}_{i=1}^m$ satisfy (WS-1) and (WS-3) for the pair $x, y \in G$, and $d_i > 0$, then \( \text{dia}(S_i) \geq 2\delta_G(z)/(C + 1) \), for all $z \in S_i$ and $1 \leq i \leq m$. Furthermore, if $d_i \geq \text{dia}(S_i)$ and $|x - y| \leq (\delta_G(x) + \delta_G(y))/2$, then there exists a constant $C' = C'(C, \alpha)$ such that

$$\delta^\alpha_G(x) + \delta^\alpha_G(y) + \sum_{k=1}^m d_i^\alpha \leq C' \text{len}_{\alpha,G}(\lambda), \quad \lambda \in \Gamma_G(x, y). \tag{2.4}$$

Proof. The first statement follows from the previous lemma by letting $c$ increase towards $1/C$. It is also easy to see that $\text{len}_{\alpha,G}(\lambda \cap B(z, \delta_G(z)/C)) \gtrsim \delta^\alpha_G(z)$ for $z \in \{x, y\}$. If $d_i \geq \text{dia}(S_i)$, then combining the first statement of the lemma with (WS-1), we see that $\text{len}_{\alpha,G}(\lambda \cap S_i) \gtrsim d_i^\alpha$. By combining these estimates for disjoint pieces of $\lambda$, we deduce (2.4). \(\square\)

Our next result carries two more lessons about slices. Ignoring a quantitatively controlled change in $C$, we may change “$d_i \geq \text{dia}(S_i)$” to “$d_i = \text{dia}(S_i)$” in the definition of an $(\alpha, C; d)$-wSlice condition, and we may assume that the $d$-diameter and Euclidean diameter of slices are comparable.

Lemma 2.5. Suppose that $0 \leq \alpha < 1$ and that $x, y$ are two points in a domain $G \subset \mathbb{R}^n$ that satisfy an $(\alpha, C; d)$-wSlice condition for some metric $d$, $d_{\mathbb{R}^n} \leq d \leq d_G$. Then there exist $(\alpha, 4C; d)$-wSlice data $\gamma, \{S_i, d_i\}_{i=1}^m$ for $x, y$ such that $d_i \leq C' \text{dia}(S_i)$ for some $C' = C'(C, \alpha)$.

Proof. It follows from the discussion after (WS-4) that there exist $(\alpha, 2C; d)$-wSlice data $\gamma, \{S_i, d_i\}_{i=1}^m$ for $x, y$ that satisfy (WS-4) (with $C$ replaced by $4C$). Any such set of data will have the property that we seek. Consider the case $\alpha = 0$. Let $\delta_i = \sup_{\lambda \cap S_i} \delta_G(z)$. Using (WS-4), we obtain $\text{len}(\gamma \cap S_i) \delta^{-1}_i \leq \text{len}_{0,G}(\gamma \cap S_i) \lesssim 1$. But by (WS-1) we have $d_i \lesssim \text{len}(\gamma \cap S_i)$ and so combining these two estimates gives the desired estimate $d_i \lesssim \text{dia}(S_i)$.

The case $\alpha > 0$ is a little trickier. Again we let $\delta_i = \sup_{\lambda \cap S_i} \delta_G(z)$. For each $k \in \mathbb{N}$, let $l_{i,k}$ be the total length of that part of $\gamma \cap S_i$ on which the distance to $\partial G$ lies in the range $(2^{-k}\delta_i, 2^{-k+1}\delta_i]$. Lemma 2.3 and the fact that $d \geq d_{\mathbb{R}^n}$ imply that $\delta_i \lesssim \text{dia}(S_i) \leq d_i$. By (WS-4), we have

$$\sum_{k=1}^\infty l_{i,k} \delta^\alpha_i 2^{-k(1-\alpha)} \lesssim \text{len}_{\alpha,G}(\gamma \cap S_i) \lesssim d_i^\alpha,$$

and so

$$l_{i,k} \lesssim \left( \frac{d_i}{\delta_i} \right)^{\alpha-1} 2^{-k(1-\alpha)} d_i \lesssim 2^{-k(1-\alpha)} d_i. \tag{2.6}$$

But by (WS-1), $\sum_{k=1}^\infty l_{i,k} = \text{len}(\gamma \cap S_i) \gtrsim d_i$. Combining this with (2.6), it follows that $l_{i,k} \gtrsim d_i$ for some $k \leq k_0 = k_0(C, \alpha)$. But then by the first half of (2.6), we see that $d_i^{1-\alpha} \lesssim \delta_i^{1-\alpha}$, and so $d_i \lesssim \delta_i \lesssim \text{dia}(S_i)$. \(\square\)
2.7. “Strong” slice domains.

Suppose $C \geq 1$ and let $d$ be a metric on $G$ satisfying $d_{\mathbb{R}^n} \leq d \leq d_G$. Then $G$ is a $(C; d)$-Slice domain if every pair $x, y \in G$ satisfies the following $(C; d)$-Slice condition: there exist a path $\gamma \in \Gamma_G(x, y)$ and pairwise disjoint open subsets $\{S_i\}_{i=1}^m$ of $G$, with $d_i \equiv \text{dia}_d(S_i) < \infty$, such that:

(i) $x \in S_0$, $y \in S_j$, and $x$ and $y$ are in different components of $G \setminus \overline{S_i}$, for all $0 < i < j$.

(ii) $\text{len}(\lambda \cap S_i) \geq d_i/C$, for all $0 < i < j$ and $\lambda \in \Gamma_G(x, y)$.

(iii) For all $t \in [0, 1]$, we have $B\left(\gamma(t), C^{-1}\delta_G(\gamma(t))\right) \subset \bigcup_{i=0}^{j} \overline{S_i}$. Also, for all $0 \leq i \leq j$, there exists $x_i \in \gamma_i$, such that $x_0 = x$, $x_j = y$, and $B\left(x_i, C^{-1}\delta_G(x_i)\right) \subset S_i$.

(iv) For all $0 \leq i \leq j$ and $z \in \gamma_i \equiv \gamma([0, 1]) \cap S_i$, we have $d_i \leq C\delta_G(z)$.

If $d$ is the Euclidean metric, we say that $G$ is a $C$-Slice domain, while if $d = d_G$, we say that $G$ is an inner $C$-Slice domain. The (Euclidean) Slice condition was first defined in [BK2], where it is assumed uniformly for all $x$ and a fixed $y$.

We now show that Slice domains are $\alpha$-wSlice domains for every $\alpha$.

Lemma 2.8. If the pair $x, y \in G$ satisfies the $(C; d)$-Slice condition for some metric $d$ satisfying $d_{\mathbb{R}^n} \leq d \leq d_G$, then for each $\alpha \in [0, 1)$, $x, y$ satisfies the $(\alpha, C'; d)$-wSlice condition for some $C' = C'(C, \alpha, n)$.

Proof. If $\gamma, \{S_i\}_{i=0}^j$ are the $(C; d)$-Slice data for the pair $x, y$, then the required $(\alpha, C'; d)$-wSlice data are $\tilde{\gamma}, \{S_i, d_i\}_{i=1}^m$, where $m = j - 1$, $d_i = \text{dia}_d(S_i)$, and $\tilde{\gamma}$ will be defined. Slice properties (ii), (iii) immediately imply (WS-1), (WS-3), so it suffices to verify (WS-2).

Properties (iii) and (iv) imply that $d_i/\delta_G(x_i) \in (1/C, C)$, and that if we write $B_z \equiv B(z, d_i/2C^2)$, then $2B_z \subset G$ for every $z \in \gamma_i$, $0 \leq i \leq j$. For fixed $i$, any pairwise disjoint subset of $\{(1/3)B_z : z \in \gamma_i\}$ has finite cardinality, with a bound dependent only on $C$ and $n$. Taking a maximal pairwise disjoint subcollection, it is clear that the associated collection of dilated balls of the form $B_z$ covers $\gamma_i$. We relabel these latter balls $B_i \equiv \{B_i^{1j}_{k(i)}\}$, where $k(i) \leq k_0 = k_0(C, n) < \infty$.

We now replace the path $\gamma$ with a polygonal path by the following finite incremental polygonalization procedure involving a partition $\{0 = s_0 < s_1 < \cdots < s_M = 1\}$ of $[0, 1]$ which we shall construct, and paths $\gamma^k$ which all have the same value at each $s_i$ and are polygonal as far as $t = s_k$. First let $s_0 = 0$, $i_0 = 0$, let $j_0$ be such that $x \in B_{100}^{j_0}$, and let $s_1$ be the largest value of $t \in [0, 1]$ such that $\gamma(t) \in B_{100}^{j_0}$. We define a new path $\gamma^1$ to be the same as $\gamma$ except that we replace the path segment $\gamma|[s_0, s_1]$ by a line segment from $\gamma(s_0)$ to $\gamma(s_1)$.

For the general inductive step, we assume that we have already defined $s_m$ and $\gamma^m$ for all $0 \leq m \leq k$. If $s_k = 1$, the procedure is declared to be complete. Otherwise, let $i_k, j_k$ be such that $\gamma^{i_k}(s_k) \in B_{ik}^{j_k}$, and let $s_{k+1}$ be the largest value of $t \in [0, 1]$ such that $\gamma(t) \in B_{ik}^{j_k}$. Predictably, we now replace the path segment $\gamma|[s_k, s_{k+1}]$ by $[\gamma(s_k) \rightarrow \gamma(s_{k+1})]$.

The balls $B_{ik}^{j_k}$ are all distinct, so this process must terminate; let $\tilde{\gamma}$ be the final, fully polygonalized, path. A straightforward calculation gives $\text{len}_{\alpha, G}(\tilde{\gamma}|[s_k, s_{k+1}]) \leq \cdots$
\[ C d_k^\alpha, \text{ where } C' = C'(C, \alpha, n), \text{ and so } \text{len}_{\alpha, G}(\gamma) \leq k_0 C' \sum_{i=0}^{m+1} d_i^2, \text{ where } m \equiv j - 1. \]

The lemma follows since \( d_0 \leq \delta_G(y) \) and \( d_{m+1} \leq \delta_G(x) \). \( \square \)

Cigar conditions always imply the corresponding slice conditions, as implied by the following slightly weakened form of [BS, Lemma 3.4].

**Lemma 2.9.** Suppose that \( 0 \leq \alpha < 1 \) and that \( G \subseteq \mathbb{R}^n \). If there is an inner \((\alpha, C)^{\prime}\)-mCigar path for the points \( x, y \in G \), then the pair \( x, y \) satisfies an inner \((\alpha, C')^{\prime}\)-wSlice condition for some \( C'' = C'(C, \alpha, n) \). If \( \alpha = 0 \), \( x, y \) also satisfies an inner \( C''^{\prime\prime}\)-Slice condition for some \( C'' = C''(C, n) \).

### 3. Hölder-type imbedding theorems

Generalizing Theorem 0.1, we shall prove the following result.

**Theorem 3.1.** Let \( 0 < \alpha < 1 < C \), \( G \subseteq \mathbb{R}^n \), and suppose that \( d \) is a metric on \( G \) satisfying \( d_{\mathbb{R}^n} \leq d \leq d_G \). Then \( G \) is an \((\alpha, C_1; d)^{\prime}\)-mCigar domain if and only if it is an \((\alpha, C_2; d)^{\prime}\)-wSlice domain which supports a \((p, C_3; d)^{\prime}\)-Hölder imbedding for \( p = (n - \alpha)/(1 - \alpha) > n \). The constants \( C_1, C_2, C_3 \) depend only on each other, and on \( \alpha \) and \( n \).

Actually, the exact type of slice condition used in this theorem does not matter, in the sense that an \((\alpha, C_1; d)^{\prime}\)-mCigar domain satisfies the strongest condition of this type (i.e., it is an inner \((\alpha, C_2)^{\prime}\)-wSlice domain), while if \( G \) satisfies the weakest condition of this type (i.e., it is an \((\alpha, C_2)^{\prime}\)-wSlice domain), then that together with an \( L^{1,p}(G, d) \hookrightarrow \text{Lip}_{1-n/p} \) imbedding implies that \( G \) is an \((\alpha, C_1; d)^{\prime}\)-mCigar domain. This stronger version of Theorem 3.1 follows by combining Lemma 2.9 with the following theorem.

**Theorem 3.2.** Let \( G \subseteq \mathbb{R}^n \), \( 0 < \alpha < 1 \), and \( p = (n - \alpha)/(1 - \alpha) > n \). Then there exists a constant \( C = C(n, \alpha) \) such that for all \( u \in L^{1,p}(G) \) and all \( x, y \in G \),

\[
|u(x) - u(y)| \leq C |d_{\alpha, G}(x, y)| + |x - y|^{\alpha - 1} \| \nabla u \|_{L_p(G)}, \quad (3.3)
\]

Conversely, if \( x, y \in G \) satisfy an \((\alpha, C_0)^{\prime}\)-wSlice condition then this inequality can be reversed for some \( u \in L^{1,p}(G) \) (dependent on \( x, y \)), and \( C = C(n, \alpha, C_0) > 0 \).

**Proof.** Throughout this proof, we write \( B_z = B(z, \delta_G(z)/2) \), for all \( z \in G \). We first prove (3.3); this proof is similar to the proof of sufficiency for Theorem 4.1(iii) in [BK2], but we include it for completeness.

Note that \( \alpha = (p - n)/(p - 1) \). If \( y \in B_z \), then (3.3) follows from the classical inequality for balls \( B \subset \mathbb{R}^n \):

\[
|u(x) - u(y)| \leq C \| x - y \|^{(p-n)/p} \| \nabla u \|_{L_p(B)}, \quad u \in L^{1,p}(B), \ x, y \in B, \quad (3.4)
\]

where \( C = C(n, p) \). For a proof of this, see [Z, 2.4.4].

Suppose therefore that \( y \notin 2B_z \). Let \( \gamma \in \Gamma_G(x, y) \) be such that \( \text{len}_{\alpha, G}(\gamma) \leq 2d_{\alpha, G}(x, y) \). We cover \( \gamma \) by the balls \( B_{\gamma(t)} \), \( 0 \leq t \leq 1 \). Note that the length of \( \gamma \cap B_{\gamma(t)} \) is at least \( \delta_G(\gamma(t))/2 \), that all points in \( B_{\gamma(t)} \) are approximately the
same distance from $\partial G$. By compactness and the Besicovitch Covering Lemma [S2, I.8.17], we can extract a subcollection $\mathcal{B} = \{B^i\}_{i=0}^j$ of $\{B_{\gamma(t)}\}$ such that $\mathcal{B}$ still covers $\gamma$ but no point in $G$ lies in more than $C = C(n)$ of the balls of $\mathcal{B}$. We arrange the indices so that there are points $\{x_i\}_{i=0}^j$ with $x_0 = x$, $x_j = y$, and $x_i \in B^{i-1} \cap B^i$ for $i = 1, \ldots, j$. By the triangle inequality, (3.4), and Hölder’s inequality, we get

$$|u(x) - u(y)| \lesssim \sum_{i=1}^j |x_i - x_{i-1}|^{(p-n)/p} \|\nabla u\|_{L^p(B^i)}$$

$$\leq \left( \sum_{i=1}^j |x_i - x_{i-1}|^{\frac{p-n}{p-1}} \right)^{p-1/p} \left( \sum_{i=1}^j \|\nabla u\|_{L^p(B^i)}^p \right)^{1/p}$$

$$\lesssim \left( \sum_{i=1}^j \int_{\gamma \cap B^i} \delta_G(z)^{\frac{p-n}{p-1}} \right)^{(p-1)/p} \|\nabla u\|_{L^p(G)}$$

$$\lesssim \left( \text{len}_{\alpha, G}(\gamma)^{(p-1)/p} \right) \|\nabla u\|_{L^p(G)}.$$
Defining \( u = \sum_{i=1}^{m} u_i \), it follows that \( u(y) - u(x) \gtrsim \sum_{i=1}^{m} d_i^{1-n/p} g_i \). Since we have not yet specified \( c_i \), we are free to choose \( g_i \) arbitrarily. Let \( g_i = cd_i^{(1-n/p)/(p-1)} \) where \( c \) is chosen so that \( \sum_{i=1}^{m} g_i^p = 1 \). It follows that \( \|\nabla u\|_{L^p(G)} = 1 \) and that

\[
  u(y) - u(x) \gtrsim \left( \sum_{i=1}^{m} d_i^{(p-n)/(p-1)} \right)^{(p-1)/p} \gtrsim (\text{len}_{\alpha, G}^{\gamma})^{(p-1)/p}.
\]

Theorem 3.2 also implies Hölder imbedding results for functions \( d \) much more general than the metrics \( d \) considered in Theorem 3.1.

**Theorem 3.5.** Suppose that \( 0 < \alpha < 1 \), \( C, \) and \( C' \) are positive constants, that \( G \subset \mathbb{R}^n \) is a bounded \((\alpha, C)\)-wSlice domain, and that \( d : G \times G \to [0, \infty) \) is such that \( d \geq C'd_{\mathbb{R}^n} \). Then \( G \) is an \((\alpha, C_1; d)\)-mCigar domain if and only if it supports an \((p, C_2; d)\)-Hölder imbedding for \( p = (n - \alpha)/(1 - \alpha) > n \). The constants \( C_1 \) and \( C_2 \) depend only on each other and on \( p, n, C, \) and \( C' \).

For example, this last theorem gives Hölder imbedding results with exponents less than the \( p \)-Hölder exponent \( 1 - n/p \) if we take \( d(x, y) = |x - y|^t \) for some \( 0 < t < 1 \); cf. [BK2, Theorem 5.2].

We can also prove imbedding results where boundedness replaces Hölder continuity. For example, we have the following analogue of Theorem 3.1.

**Theorem 3.6.** Suppose that \( 0 < \alpha < 1 \) and that \( G \subset \mathbb{R}^n \) is a bounded domain with \( x_0 \in G \). Then the \( d_{\alpha, G} \)-diameter of \( G \) is at most \( C_1 \) if and only if the pair \( x, x_0 \) satisfies an \((\alpha, C_2)\)-wSlice condition for all \( x \in G \), and \( G \) supports the \( L^\infty \) imbedding

\[
|u(x) - u(x_0)| \leq C_3 \|\nabla u\|_{L^p(G)}, \quad u \in L^{1,p}(G).
\]

The constants \( C_1, C_2, C_3 \) depend only on each other, and on \( x_0, \alpha, \) and \( n \).

**Proof.** The \( d_{\alpha, G} \)-boundedness of \( G \) implies an \( \alpha \)-wSlice condition for \( x, x_0 \) with zero slices and, in view of Theorem 3.2, it also implies (3.7). Theorem 3.2 also implies the converse direction.

We next wish to discuss the imbedding \( W^{1,p}(G) \hookrightarrow \text{Li}_{1-n/p}(G, d), \quad p > n \). When \( d = d_{\mathbb{R}^n} \) and \( G \) is bounded, this is equivalent to the imbedding \( L^{1,p}(G) \hookrightarrow \text{Li}_{1-n/p,e}(G, d) \); see [KR]. We shall show that this equivalence and others extend to imbeddings defined in terms of any of a large class of functions \( d \). The proof of equivalence, an adaptation of the methods of Kőskela and Reitich [KR] for the Euclidean metric, is independent of a relationship between the Sobolev and Lipschitz exponents; however, since \( d \) is allowed to be quite general, this decoupling is only a convenience and not a generalization. With these equivalences in hand, we can use our earlier methods to find conditions for a domain to support these imbeddings.

We define the variational \( p \)-capacity \( \text{cap}_{p}(E, F; G) \), where \( E, F \) are compact subsets of \( G \). First let \( L(E, F; G) \) denote the class of all functions \( u \in L^{1,p}(G) \) that
are continuous on $G \cup E \cup F$ and equal $C_0$ on $E$ and $C_1$ on $F$, for some numbers $C_0, C_1$ satisfying $|C_1 - C_0| = 1$. Then

$$\text{cap}_p(E, F; G) = \inf_{u \in L(E, F; G)} \int_G |\nabla u|^p.$$ 

We get an equivalent definition of $\text{cap}_p(E, F; G)$ if we replace $L(E, F; G)$ by its subset $L_0(E, F; G)$ consisting of those functions whose values lie between $C_0 = 0$ and $C_1 = 1$; see [Mz, 4.1.1]. We abbreviate singleton sets $\{x\}$ to $x$ when dealing with capacities.

**Theorem 3.8.** Let $G \subset \mathbb{R}^n$ be a domain and let $s \in (0, 1)$, $p \in (n, \infty)$. Suppose that $d : G \times G \to [0, \infty)$ satisfies $\psi(|x - y|) \leq d(x, y) \leq \phi(|x - y|)$, where $\psi, \phi : (0, \infty) \to (0, \infty)$ are non-decreasing functions and $\lim_{t \to 0^+} \phi(t) = 0$. Then the following are equivalent:

(i) $W^{1,p}(G) \hookrightarrow C^{0,s}(G, d)$;
(ii) $W^{1,p}(G) \hookrightarrow \text{Lip}_s(G, d)$;
(iii) $W^{1,p}(G) \hookrightarrow \text{Lip}_{s, \epsilon_1}(G, d)$ for some $\epsilon_1 > 0$;
(iv) $L^{1,p}(G) \hookrightarrow \text{Lip}_{s, \epsilon_2}(G, d)$ for some $\epsilon_2 > 0$;
(v) There exists some $\epsilon_3 > 0$ such that $\text{cap}_p(x, y; G) \geq Cd(x, y)^{-ps}$ whenever $x, y \in G, |x - y| \leq \epsilon_3$.

If $G$ is bounded, (i)–(v) above are also equivalent to

(vi) $L^{1,p}(G) \hookrightarrow \text{Lip}_s(G, d)$.

Furthermore, the various imbedding constants, and the numbers $\epsilon_i$ depend only on each other, $\phi, \psi, s, p, n$, and (if (vi) is the implied condition) $\text{diam}(G)$.

We shall need three lemmas, the first of which is due to Maz’ya [Mz, 5.1.1].

**Lemma 3.9.** Suppose $p \in (n, \infty)$ and $G \subset \mathbb{R}^n$ is a domain. Then $W^{1,p}(G) \hookrightarrow L^{\infty}(G)$ if and only if there exist numbers $r, C > 0$ such that $\text{cap}_p(x, \overline{G \setminus B(x, r)}; G) \geq C$ for all $x \in G$. Moreover the imbedding norm and the constants $r, C$ depend only on each other, $p, n$.

**Lemma 3.10.** Suppose $p \in (n, \infty)$, $0 < \epsilon \leq \infty$, that $G \subset \mathbb{R}^n$ is a domain, and that $d : G \times G \to [0, \infty)$ is positive off the diagonal. Then $L^{1,p}(G) \hookrightarrow \text{Lip}_{s, \epsilon}(G, d)$ with imbedding constant $C$ if and only if $\text{cap}_p(x, y; G) \geq C^{-p}d(x, y)^{-ps}$ whenever $x, y \in G, |x - y| \leq \epsilon$.

**Proof.** The fact that the capacity condition follows from the imbedding is obvious. For the converse, assume that $u \in L^{1,p}(G)$ with $u(x) \neq u(y)$ for some $x, y \in G, |x - y| < \epsilon$. Applying the capacity condition to the function $z \mapsto u(z)/|u(x) - u(y)|$, the imbedding follows. \qed

**Lemma 3.11.** Let $G$, $s$, $p$, and $\phi$ be as in Theorem 3.8, and let $d : G \times G \to [0, \infty)$ be positive off the diagonal and satisfy $d(x, y) \leq \phi(|x - y|)$. If for some $\epsilon > 0$, $W^{1,p}(G) \hookrightarrow \text{Lip}_{s, \epsilon}(G, d)$ with imbedding constant $C$ then for some $\epsilon' > 0$,
$L^{1,p}(G) \hookrightarrow \text{Lip}_{s,e'}(G,d)$ with imbedding constant $2C$. The number $e'$ depends only on $C\phi^s$, $e$, $p$, and $n$.

Proof. Let $x, y \in G$ satisfy $|x - y| \leq \epsilon$, and fix $\eta \in C_0^\infty(\mathbb{R}^n)$ such that $\text{supp} \eta \subset B(0, \epsilon)$, $\eta(0) = 1$, $|\nabla \eta| \leq 2/\epsilon$, and $0 \leq \eta \leq 1$. Taking $u \in L_0(x, y; G)$, we see that $v(z) = u(z)\eta(z - y)$ defines a function in $W^{1,p}(G)$ and that

$$1 = v(y) - v(x) \leq C^p d(x, y)^{ps} \int_G |v|^p + |\nabla v|^p$$

$$\leq C^p d(x, y)^{ps} \int_G [\chi_{B(y, \epsilon)} + 2^{p-1} (|\nabla \eta|^p + |\nabla u|^p)]$$

$$\leq C^p d(x, y)^{ps} \left[ C_0 + 2^{p-1} \int_G |\nabla u|^p \right],$$

where $C_0 = C_0(\epsilon, p, n)$. Thus

$$C^{-p} d(x, y)^{-ps} - C_0 \leq 2^{p-1} \text{cap}_p(x, y; G).$$

Choosing $\epsilon' > 0$ so small that $|C\phi(e')^s|^{-p} \geq 2C_0$, we see that if $x, y \in G$, $|x - y| \leq \epsilon'$, then $\text{cap}_p(x, y; G) \geq (2Cd(x, y)^s)^{-p}$. Hence Lemma 3.10 yields the claim. □

Proof of Theorem 3.8. It follows immediately from the definitions that (i) implies (ii), and (ii) implies (iii). By Lemma 3.11, (iii) implies (iv), and by Lemma 3.10, (iv) implies (v). We next show that (v) implies (i). First it follows from Lemma 3.9 that $W^{1,p}(G) \hookrightarrow L^\infty(G)$, so it suffices to show that $W^{1,p}(G) \hookrightarrow \text{Lip}_s(G)$. By Lemma 3.10, we have $W^{1,p}(G) \hookrightarrow \text{Lip}_{s, \epsilon_3}(G, d)$. Let $u \in W^{1,p}(G)$, and $x, y \in G$ with $|x - y| \geq \epsilon_3$. Since $W^{1,p}(G) \hookrightarrow L^\infty(G)$, we have

$$|u(x) - u(y)| \leq 2\|u\|_{L^\infty(G)} \lesssim \|u\|_{W^{1, p}(G)} \lesssim \psi(\epsilon_3)^{-s} d(x, y)^s \|u\|_{W^{1, p}(G)},$$

and (i) follows since $\psi(\epsilon_3)^{-s} \lesssim 1$.

We have shown that (i)-(v) are equivalent. It is clear that (vi) implies (iv). To finish the theorem, we prove (vi) under the assumptions that $\text{dia}(G) \equiv \Delta < \infty$ and that (iv) holds with imbedding constant $C$. By a standard covering lemma (see [S1, L6, I7]) it follows that $G$ can be covered by a countable collection of balls $\{B_i\}_{i=1}^N$ of radius $\epsilon_2$ such that the smaller balls $\{(1/5)B_i\}_{i=1}^N$ are pairwise disjoint and the centers of these balls lie in $G$. Since these smaller balls are disjoint and all lie in an $\epsilon_2$-neighbourhood of $G$, we must have $N \leq (5(\Delta + \epsilon_2)/\epsilon_2)^p$. By chaining the triangle inequality, we see that $|u(x) - u(y)| \leq C N \phi(\epsilon_2)^s \|\nabla u\|_{L^p(G)}$, whenever $x, y \in G$. This gives the desired imbedding inequality whenever $|x - y| \geq \epsilon_2$. Since we already know that the inequality holds when $|x - y| < \epsilon_2$, we are done. □

It is easy to apply Theorem 3.8 to get versions of our earlier imbedding theorems for the imbedding $W^{1,p}(G) \hookrightarrow \text{Lip}_{1-n/p}(G, d)$. For instance, we shall state without proof a $W^{1,p}(G)$ version of Theorem 3.5 after some preliminary definitions.

Let $G \subseteq \mathbb{R}^n$. We say that $G$ is $(\alpha, C, \epsilon)$-wSlice domain, or an (inner) $(\alpha, C_1, \epsilon; d)$-mCigar domain, if every pair of points $x, y \in G$ with $|x - y| \leq \epsilon$ satisfies an $(\alpha, C)$-wSlice condition, or an (inner) $(\alpha, C; d)$-mCigar condition, respectively. If we do not care about the value of $\epsilon$, we simply refer to local $(\alpha, C)$-wSlice and local (inner) $(\alpha, C; d)$-mCigar domains; similarly, we can define local (inner) uniform domains. It is easy to see that a local (inner) uniform domain is a local (inner) $\alpha$-mCigar domain for every $\alpha \in (0, 1)$.
Theorem 3.12. Suppose that $0 < \alpha < 1$, $p = (n - \alpha)/(1 - \alpha) > n$, $\epsilon$, $C$, and $C'$ are positive constants, that $G \subset \mathbb{R}^n$ is a bounded $(\alpha, C, \epsilon)$-wSlice domain, and that $d : G \times G \to [0,\infty)$ is such that $C'|x - y| \leq d(x, y) \leq \phi(|x - y|)$ for some non-decreasing function $\phi : (0,\infty) \to (0,\infty)$ satisfying $\lim_{t \to 0^+} \phi(t) = 0$. Then $G$ is an $(\alpha, C_1, \epsilon_1; d)$-mCigar domain for some $\epsilon > 0$ if and only if $W^{1,p}(G) \hookrightarrow \text{Lip}_{1-n/p}(G, d)$ with imbedding constant $C_2$. The constants $C_1$, $C_2$, $\epsilon_1$ depend only on each other and on $C$, $C'$, $\phi$, $\alpha$, $\epsilon$, $p$, and $n$.

4. IMBEDDING EXAMPLES AND NON-EXAMPLES

We first apply the theorems in the last section to some specific domains. For notational convenience, we deal only with planar regions, but these examples can readily be generalized to higher dimensions; we denote the dimension in imbeddings by “$n$” as usual, rather than “2” to emphasise this. In all examples, we assume that $p \in (2,\infty)$ and look at imbeddings of the form $X \subset \text{Lip}_{1-n/p}(G, d)$ for various functions $d$, where $X$ is either $L^{1,p}(G)$ or $W^{1,p}(G)$. Every domain we consider is simply-connected and thus an $\alpha$-wSlice domain for each $\alpha \in [0,1)$ by the Riemann mapping theorem together with Lemmas 2.8 and 2.9.

Example 4.1. For each $k \in \mathbb{N}$, let $U_k$ be the roughly U-shaped region defined by

$$U_k = (0,1) \times [0,2) \cup (0,4) \times (1,2) \cup (3,4) \times (2^{-k},2)$$

and let $L_k$ be the affine map defined by $L_k(z) = 2^{-k-3}z + (2^{-k},1)$. Next let $H_k = L_k(U_k)$ and $G = (0,1)^2 \cup (\bigcup_{k=1}^\infty H_k)$, so that $G$ is a square with a sequence of almost-closed hooks attached. It is easy to verify that $G$ is an inner uniform domain and hence an inner $\alpha$-mCigar domain for every $\alpha \in [0,1)$. Since $|x - y| \leq d_G(x,y) \leq 2|x - y|^{1/2}$, it follows from Theorem 3.5 that $L^{1,p}(G) \hookrightarrow \text{Lip}_{1-n/p}(G, d_G)$.

If $\alpha \in (0,1)$, then $d_{\alpha,G}(x,y) \leq C_{\alpha}|x - y|^{\alpha/2}$, with approximate equality for certain pairs of points $x,y$ with $|x - y|$ is arbitrarily small (to see this, pick $x \in H_k$ and $y \in (0,1)^2$ to be points near the boundary of $G$ on either side of the narrow gap between $H_k$ and $(0,1)^2$). Thus we have $L^{1,p}(G) \hookrightarrow \text{Lip}_{1-n/p}(G, d_{\alpha, G})$ only for $t \leq 1/2$; this is weaker than the previous inner metric imbedding since $d_G \leq 2d_G^{1/2}$. By Theorem 3.8, we cannot increase $t$ even if we replace $L^{1,p}(G)$ by $W^{1,p}(G)$.

Example 4.2. Let $U_k$ be as in Example 4.1, but now let $L_k(z) = 4^kz + (4^{k+3},0)$, $H_k = L_k(U_k)$, and $G = [\mathbb{R} \times (-\infty,0)] \cup (\bigcup_{k=1}^\infty H_k)$. As before, $G$ is inner uniform and $L^{1,p}(G) \hookrightarrow \text{Lip}_{1-n/p}(G, d_G)$. Writing $d_t(x,y) = |x - y| \vee |x - y|^t$, we have $L^{1,p}(G) \hookrightarrow \text{Lip}_{1-n/p}(G, d_t)$ only when $t \geq 2$; note that $d_t$ is not even a metric. This imbedding is again weaker than the inner metric imbedding. By contrast with Example 4.1, replacing $L^{1,p}(G)$ by $W^{1,p}(G)$ makes quite a difference. In fact, $G$ is a local uniform domain (with $\epsilon = 1$), and so $W^{1,p}(G) \hookrightarrow \text{Lip}_{1-n/p}(G, d_{\mathbb{R}^2})$.

Example 4.3. Let $G = \mathbb{R}^2 \setminus [0,\infty) \times [-1,1]$. Then $G$ is both a local uniform and an inner uniform domain, so $W^{1,p}(G) \hookrightarrow \text{Lip}_{1-n/p}(G, d_{\mathbb{R}^2})$ and $L^{1,p}(G) \hookrightarrow \text{Lip}_{1-n/p}(G, d_G)$. However $L^{1,p}(G) \not\hookrightarrow \text{Lip}_{1-n/p}(G, d_{\mathbb{R}^2})$ since $G$ is not an $\alpha$-mCigar domain for any $\alpha \in [0,1)$.

The examples so far all support the imbedding $L^{1,p}(G) \hookrightarrow \text{Lip}_{1-n/p}(G, d_G)$. To get examples for which this imbedding fails, we simply need some form of a cusp or long corridor in our domain.
Example 4.4. Let $G = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < x_2 < x_1^{1/2}\}$. $G$ is convex, so $d_G = d_{G_2}$. Let $\alpha = (p-n)/(p-1)$, and for $t > 1$, let $d_t = d_G \vee d_G^t$. A little calculation shows that the minimal exponent for which we have $d_{\alpha,G} \leq C d_{G_2}^t$ is $t = (1 + \alpha)/2\alpha$; note that the minimum is achieved for a pair of points $z = (1, 0)$, $w = (w_1, 0)$, with $w_1 \to \infty$. Thus by Theorem 3.5, $L^{1,p}(G) \hookrightarrow \text{Lip}_{1-n/p}(G, d_{(1+\alpha)/2\alpha})$, but in light of Theorem 0.1, $L^{1,p}(G) \not\hookrightarrow \text{Lip}_{1-n/p}(G, d_G)$. However, $G$ is a local uniform domain, so Theorem 3.12 implies that $W^{1,p}(G) \hookrightarrow \text{Lip}_{1-n/p}(G, d_G)$.

Our next proposition provides specific settings in which the strong slice condition is much more geometrically restrictive than any $w$Slice condition.

Proposition 4.5. Fix numbers $\alpha, c \in (0, 1)$. Let $t_j \in (0, 1/2)$ with $t_{j+1} < ct_j$ for each $j \in \mathbb{N}$. Let $T$ be the planar triangle $\{(x, y) : 0 < x < 1, |y| < x\}$ and let $F = \bigcup_{j=1}^{\infty} F_j$, where $F_j$ consist of the $n_j$ points which divide the line segment $(t_j, y) : |y| < t_j$ into $n_j + 1$ equal subsegments. Then $G = T \setminus F$ is a quasiconvex (inner) $\alpha$-mCigar domain, and hence an (inner) $\alpha$-$w$Slice domain, but is an (inner) $\alpha$-Slice domain if and only if the sequence $(n_j)$ is bounded.

Proof of Proposition 4.5. It is clear that $G$ is quasiconvex, so every inner slice condition is equivalent to its Euclidean counterpart. If $(n_j)$ is bounded then clearly $G$ is a uniform domain, and hence a Slice domain. Suppose $(n_j)$ is unbounded. We shall show that $G$ cannot even satisfy a uniform Slice condition for $z, z_0$, where $z_0 = (3/4, 0)$ and $z = ((t_k + t_{k+1})/2, 0)$ for arbitrary $k \in \mathbb{N}$. If a $C$-Slice condition holds then the path $\gamma$ for the pair $z, z_0$ meets the line $x = t_k$ at some point $z'$. Now $z'$ lies in some slice $S_i$. We must have $\text{dia} S_i \gtrsim t_k/C$: this follows from part (iv) of the Slice definition if $i = 0$ or $i = j = j(k)$, and from the fact that $S_i$ separates $z$ and $z_0$ for all other $i$. On the other hand, $\delta_G(z') \leq t_k/n_k$. Thus, part (iv) of the Slice definition implies that $n_j \lesssim C^2$, as required. The converse direction is easy.

It remains to show that $G$ is an $\alpha$-mCigar domain. We first wish to construct auxiliary subdomains $G_k$, $k \in \mathbb{N}$. Let $L_{k,i}$ and $z_{k,i}$, $i = 0, \ldots, n_k$, be a enumeration of the (line segment) components of $G \cap (\{t_k\} \times \mathbb{R})$ and their midpoints, let $D_{k,i}$ be the intersection of the strip $\{(u, v) : t_k + t_{k+1} < 2u < t_k + t_{k-1}\}$ with the disk that has $L_{k,i}$ as a diameter, and let

$$G_k = \{(u, v) \in G : t_k < u < t_{k-1}\} \cup \left( \bigcup_{i=0}^{n_k} D_{k,i} \right).$$

It is not hard to show that $G_k$ is a $C$-uniform domain for some universal constant $C$, and so $G_k$ is an $(\alpha, C')$-mCigar domain for some $C' \lesssim 1$.

If a pair of points $z_i = (x_i, y_i)$, $i = 1, 2$, does not lie in a single subdomain $G_k$, we can deduce an $\alpha$-mCigar condition for them by combining the conditions for intermediate sets of the form $G_k$. To see this, we assume without loss of generality that $t_{j+1} < x_2 \leq t_j \leq t_i < x_1 \leq t_{i-1}$ for some $i < j$. For each $k$ satisfying $i - 1 \leq k \leq j + 1$, we choose points $w_k$ such that $w_{i-1} = z_1$, $w_{j+1} = z_2$, and $w_k$ is one of the points $z_{k,i}$ for each $i \leq k \leq j$. If $i < k < j$, it does not matter which of these points we pick but we pick $w_i$ and $w_j$ so as to minimize $|w_{i+1} - w_i|$ and $|w_{j+1} - w_j|$. The desired $\alpha$-mCigar condition follows from the triangle inequality and the geometric decay of the $d_{\alpha,G}$-distances from $w_k$ to $w_{k+1}$. □
Finally, we consider the class of *Steiner symmetric cusp domains*

\[ SC(\phi, n) \equiv \{ x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1} : \|x\| < \phi(x_1), 0 < x_1 < 1 \}, \]

where \( \phi : (0, 1) \to (0, \infty) \) is a strictly increasing continuous function which satisfies \( \phi(0) = 0 \) and \( \liminf_{t \to 0^+} \phi(t)/t = 0 \). We shall see that these cusp domains never support \( p \)-Hölder imbeddings. Note that if instead the above \( \liminf \) were positive and \( |\phi(t_1) - \phi(t_2)| > |t_2 - t_1| \) for all \( 0 < t_1, t_2 < t_0 \), then the boundary of the domain \( SC(\phi, n) \) would locally be the graph of a Lipschitz function and thus all of the Hölder imbeddings would be valid on \( SC(\phi, n) \) by the classical theorem (which, by the way, is due to Morrey [Mo]).

**Proposition 4.6.** Suppose that \( U = SC(\phi, n) \) is a Steiner symmetric cusp domain and that \( p > n \). Then \( U \) does not support an (inner) \( p \)-Hölder imbedding, while \( B(0, 2) \setminus U \) supports a \( p \)-Hölder imbedding if and only if \( n > 2 \), and it supports an inner \( p \)-Hölder imbedding in all dimensions.

The fact that cusp domains do not support Hölder imbeddings is not very new (for instance it is implicit in [KR, 5.2]), but our approach is different.

**Proof of Proposition 4.6.** Let \( \alpha = (p-n)/(p-1) \). We first consider the non-imbedding result for \( U \) itself. We claim that pairs of points on the \( x_1 \)-axis of \( U \) satisfy an \((\alpha, C)\)-wSlice condition for some \( C = C(p, n) \). If \( n = 2 \), then \( U \) is conformally equivalent to the unit disk; the claim then follows from Lemma 2.9. If \( n > 2 \), then the associated planar domain \( U_2 = SC(\phi, 2) \) is an \( \alpha \)-wSlice domain. Take arbitrary points \( x = (x_1, 0) \) and \( y = (y_1, 0) \in U_2 \), and let \( \gamma, \{S_i, e_i\}_{i=1}^m \) be the associated slice data. By Lemma 2.5 we may assume that \( d_i = \text{dia}(S_i) \). We may also assume that \( \gamma = [x \to y] \), since this is the minimal \( d_{\alpha, U_2} \)-length path for the pair \( x, y \). Let \( X, Y \in U \) be defined by \( X = (x_1, 0) \) and \( Y = (y_1, 0) \); of course “0” is now an \((n-1)\)-dimensional vector. The slice data for \( x, y \) induce slice data \( \gamma', \{T_i, e_i\}_{i=1}^m \) for \( X, Y \): take \( \gamma' = [X \to Y], T_i = \{(u_1, u') : (u_1, |u'|) \in S_i \} \), and \( e_i = \text{dia}(T_i) \). Since each \( S_i \) intersects the \( x_1 \)-axis, it is not hard to see that \( e_i \approx d_i \). With this in hand, it is a routine matter to verify the slice properties.

In light of the \( \alpha \)-wSlice property that we have proved and the converse part of Theorem 3.2, we need only check that pairs of points on the \( x_1 \)-axis of \( U \) can be found that fail to satisfy any given \((\alpha, C)\)-mCigar condition. Consider points \( x = (a, 0), y = (b, 0) \), where \( 0 < a < b < 1 \). For any \( z = (z_1, 0) \), we have \( \delta_U(z) \leq \phi(z_1) \). If \( x, y \) satisfy an \((\alpha, C)\)-mCigar condition, then

\[
F(b) \equiv \int_0^b \phi(t)^{\alpha-1} dt \leq Cb^\alpha, \tag{4.7}
\]

for each \( b \in [0, 1] \). Next, let \( M > 0 \) be arbitrary. By the assumptions on \( \phi \), there exists \( t_M > 0 \) such that \( \phi(t) < t_M/M \) on the entire interval \([0, t_M]\). Putting this into (4.7) would yield \( M^{1-\alpha}t_M^\alpha \leq Ct_M^\alpha \), which is certainly untenable for large \( M \).

Next we consider \( G \equiv B(0, 2) \setminus U \). As the reader may verify, \( G \) is an inner uniform domain in all dimensions, and is a uniform domain if \( n > 2 \); the positive imbedding results follow. In the planar case, \( G \) is not an \( \alpha \)-mCigar domain, as can be verified by considering two points \((x_1, \pm 2\phi(x_1))\). But \( G \) an (inner) \( \alpha \)-wSlice domain (since it is inner uniform), and so \( G \) cannot support a \( p \)-Hölder imbedding when \( n = 2 \).  \( \square \)
References


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