WEAK SLICE CONDITIONS AND HÖLDER IMBEDDINGS

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Abstract. We introduce weak slice conditions and investigate imbeddings of Sobolev spaces in various Lipschitz-type spaces.

0. Introduction

Bojarski [B] proved that Sobolev-Poincaré imbeddings are valid on all John domains; see also [Mto]. In [BK1], it is shown that John domains are essentially the right class for this imbedding, since a bounded domain $G \subset \mathbb{R}^n$ is a John domain if and only if it supports a Sobolev-Poincaré imbedding and satisfies a certain *separation condition*. Corresponding results for the $p = n$ (Trudinger) and $p > n$ (Hölder) cases of the Sobolev Imbedding Theorem are given in [BK2], where it is shown that for domains satisfying a certain *slice condition*, each of these imbeddings is equivalent to a *mean cigar condition* dependent on p; see Section 1 for definitions of these concepts. For other results on Hölder imbeddings, we refer the reader to [A], [Mz], and [KR].

In one way, the results of [BK2] are less satisfying than those of [BK1]. In [BK1], the strong geometric condition (John) is equivalent to the combination of the weak geometric condition (separation) and a Sobolev-Poincaré imbedding. However in [BK2], the weak geometric condition (slice) is not implied by the strong geometric condition (mean cigar) for any value of $p \geq n$. For $p = n$, Buckley and O'Shea [BO] overcame this deficiency by showing that the strong geometric condition is equivalent to the combination of a so-called *weak slice condition* (which is implied by a slice condition) and the Trudinger imbedding. Here we prove the following analogous result for $p > n$; the terminology is explained in Sections 1 and 2.

Theorem 0.1. *Let $0 < \alpha < 1$ and $G \subsetneq \mathbb{R}^n$. Then G is an (inner)*[1] *α-mCigar domain if and only if it is an (inner) α-wSlice domain which supports an (inner) p-Hölder imbedding for $p = (n - \alpha)/(1 - \alpha) > n$.*

More generally, we prove variations of Theorem 0.1 with the Euclidean metric replaced by some other metric (for instance the inner Euclidean metric), and we

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[1] A statement with one or more parenthesized instances of "inner" is meant to be expanded into two statements: one includes "inner" in all locations, and the other excludes it everywhere.

investigate related imbeddings of $W^{1,p}(G)$ in Lipschitz spaces with respect to one of a large class of metrics (and even some non-metrics), generalizing results of [KR]. Reflecting the fact that the strong geometric condition is genuinely different for each $p \geq n$, our weak slice conditions (all weaker than the slice condition of [BK2]) will also be genuinely different for each value of α.

The rest of the paper is organized as follows. After some preliminaries, we introduce the slice conditions in Section 2, where we also prove some basic related results. The imbedding theorems are stated and proved in Section 3, and finally we look at some specific examples in Section 4.

We note that the weak slice theory developed here is used in [BS] to investigate what product domains are quasiconformal images of balls or other nice domains.

1. Preliminaries

1.1. Notation.

We adopt two common conventions. First, we drop parameters if we do not wish to specify their values; for instance, we define C-uniform domains, but often talk about uniform domains. Second, we write $C = C(x, y, \dots)$ to mean that a constant C depends only on the parameters $x, y, \dots$.

If $S \subset \mathbb{R}^n$ is measurable, then $|S|$ is the Lebesgue measure of S, and u_S is the average value of a function u on S. H^k denotes k-dimensional Hausdorff measure. In proofs, we write $A \lesssim B$ if $A \leq CB$ for some constant C dependent only on allowed parameters; we write $A \approx B$ if $A \lesssim B \lesssim A$. We write $A \wedge B$ and $A \vee B$ for the minimum and maximum, respectively, of the quantities A and B. Unless otherwise stated, G is a proper subdomain of $\mathbb{R}^n$.

For this paragraph and the next, $U \subset \mathbb{R}^n$. Given $x, y \in U$, we define $\delta_U(x)$ to be the distance from x to ∂U, and $\Gamma_U(x, y)$ to be the class of rectifiable paths $\lambda : [0, t] \to U$ for which $\lambda(0) = x$ and $\lambda(t) = y$. If γ is a rectifiable path in U, and $\alpha \in \mathbb{R}$, and ds is arclength measure, we define

$$\mathrm{len}_{\alpha,U}(\gamma) = \int_\gamma \delta_U^{\alpha-1}(z)\, ds(z),$$

together with an associated metric

$$d_{\alpha,U}(x, y) = \inf_{\gamma \in \Gamma_U(x,y)} \mathrm{len}_{\alpha,U}(\gamma), \qquad x, y \in U.$$

Of course, $d_{\alpha,U}(x, y) = \infty$ unless x, y lie in the same path component of U. We write len in place of $\mathrm{len}_{1,U}$; note that $d_{1,U}$ is the *inner Euclidean metric*. For the sake of brevity, it is convenient to abuse notation by, for instance, writing $\mathrm{len}_{\alpha,U}(\gamma \cap S)$ for the $d_{\alpha,U}$-length of those parts of a path γ lying in a subset S of U. We write $[x, y]$ for the line segment joining a pair of points in $\mathbb{R}^n$, $[x \to y]$ for the path parametrized by arclength that goes from x to y along $[x, y]$.

We are mainly interested in $d_{\alpha,U}$ when $\alpha \in [0, 1]$ and U is a domain. When U is a domain and $\alpha \leq 0$, $d_{\alpha,U}$-geodesics exist for every pair of points; see [GO], [Mtn]. However, they can fail to exist for any choice of $\alpha \in (0, 1]$. We cannot find a reference for this fact, so we now pause to give a counterexample (the case $\alpha = 1$ is of course trivial).

Example 1.2. The desired domain will consist of the unit disk $B = B(0,1)$ with certain segments of the real axis being removed. Let $z = (0, 1/2)$ and $G_0 \equiv B \setminus [(-t,0),(t,0)]$, where $0 < t < 1$ is so close to 1 that $d \equiv d_{\alpha,G_0}(z,-z)$ is strictly larger than the value $d_\star$ of the (convergent improper integral which defines the) d_{α,G_0}-length of the linear segment $[z,-z]$. Next, let $(x_k)_{k=1}^\infty$ be a strictly decreasing sequence with limit zero and $x_1 < t$. We write $z_k = (x_k, 0) \in B$, $h_1 = t - x_1$, and $h_k = x_{k-1} - x_k$, $k > 1$. Let $d_{0,j}$ denote the infimum of the (convergent improper integrals which define the) d_{α,G_0}-lengths of paths from z to $-z$ that pass through z_j, $j \in \mathbb{N}$. By taking x_1 to be small enough, we may assume that $d_{0,1} < d$.

We define $I_k = (z_k - (\epsilon_k, 0), z_k + (\epsilon_k, 0))$, $k \in \mathbb{N}$, and $G_m = G_0 \cup (\bigcup_{k=1}^m I_k)$ for each $k \in \mathbb{N} \cup \{\infty\}$; the desired domain G will be G_∞. Here ϵ_k is positive, less than $(h_k \wedge h_{k+1})/2$ (so that the intervals I_k are disjoint), and so small that

$$d > d_{k,1} > d_{k,2} > \cdots > d_{k,j} > \cdots > d_\star, \qquad k \in \mathbb{N} \tag{1.3}$$

and

$$(1 - (k+1)^{-2}) \le \frac{d_{k,j} - d_\star}{d_{k-1,j} - d_\star} \le 1, \qquad k \in \mathbb{N} \tag{1.4}$$

where for each $j \in \mathbb{N}$, $k \in \mathbb{N} \cup \{\infty\}$, $d_{k,j}$ is the infimum of the d_{α,G_k}-lengths of paths from z to $-z$ that pass through I_j (if $j \le k$) or through z_j (if $j > k$). We leave to the reader the task of verifying that these inequalities are satisfied for appropriate $\epsilon_k > 0$ and that (1.4) implies that $\lim_{k\to\infty} d_{k,j} = d_{\infty,j} > d_\star$. It is also not hard to see that $d_{\alpha,G_\infty}(z,-z) = d_\star$, but that there is no geodesic.

Given $x \in U$, $E, F \subset U$, and a metric ρ on U, we write $d_\rho(E,F)$ for the ρ-distance between E and F, $\mathrm{dia}_\rho(E)$ for the ρ-diameter of E, and $B_\rho(x,r) = \{y \in U : d_\rho(x,y) < r\}$. If $\rho = d_{1,U}$, we instead write $d_U(E,F)$, $\mathrm{dia}_U(E)$, and $B_U(x,r)$ for these concepts, while if ρ is the Euclidean metric (and so $U = \mathbb{R}^n$), we write $d(E,F)$, $\mathrm{dia}(E)$, and $B(x,r)$. For brevity, we define $d_U = d_{1,U}$; in particular, $d_{\mathbb{R}^n}$ is the Euclidean metric. Note that distance to the boundary of U is the same with respect to $d_{\mathbb{R}^n}$ and d_U, and that $B_U(x,r) = B(x,r)$ if $r \le \delta_U(x)$.

1.5. Function spaces and Hölder-type imbeddings.

For $n < p < \infty$, $L^{1,p}(G)$ is the space of functions $f : G \to \mathbb{R}$ with distributional gradients in $L^p(G)$, and $W^{1,p}(G) = L^p(G) \cap L^{1,p}(G)$ is the corresponding Sobolev space. We write $\|u\|_{L^{1,p}(G)} = \|\nabla u\|_{L^p(G)}$ and $\|u\|_{W^{1,p}(G)} = \|u\|_{L^p(G)} + \|u\|_{L^{1,p}(G)}$.

Let $0 < t \le 1$, $0 < \epsilon \le \infty$, and let $d : G \times G \to [0,\infty)$ be a function which is positive off the diagonal. We define $\mathrm{Lip}_{t,\epsilon}(G,d)$ to be the space of all functions $u : G \to \mathbb{R}$ for which

$$\|u\|_{\mathrm{Lip}_{t,\epsilon}(G,d)} \equiv \sup_{\substack{x,y\in G \\ 0<|x-y|\le\epsilon}} \frac{|u(x) - u(y)|}{d(x,y)^t} < \infty.$$

If $d(x,y) = |x-y|$, we simply write $\mathrm{Lip}_{t,\epsilon}(G)$; note that $\mathrm{Lip}_{t,\epsilon}(G,d)$ is the familiar space of functions which are Hölder continuous of order t. If $\epsilon \ge \mathrm{dia}(G)$, we omit the "$\epsilon$" subscript. We also define $C^{0,t}(G,d)$ to be the space of all bounded functions in $\mathrm{Lip}_t(G,d)$ and write

$$\|u\|_{C^{0,t}(G,d)} = \|u\|_{L^\infty(G)} + \|u\|_{\mathrm{Lip}_t(G,d)}.$$

The notation $C^{0,t}(G,d)$ is not very appropriate unless we at least have $d(x,y) \to 0$ as $x \to y$, but we do not wish to put restrictions on d at this point.

For any pair of these spaces, we write $A \hookrightarrow B$ if $A \subset B$ and $\|\cdot\|_B \le C\|\cdot\|_A$. We call the smallest constant C for which this condition is valid the *imbedding constant for* $A \hookrightarrow B$. Of course this quantity and the various quantities $\|\cdot\|_A$ are not in general "norms" in the functional analysis sense, but the notation is still convenient. We are particularly interested in the imbedding $L^{1,p}(G) \hookrightarrow \mathrm{Lip}_{1-n/p}(G,d)$, and say that G *supports the* $(p,C;d)$ *Hölder imbedding* if this imbedding holds with imbedding constant C. In particular, we say that G *supports the* (p,C)*-Hölder imbedding* (or *inner* (p,C)*-Hölder imbedding*) if $d = d_{\mathbb{R}^n}$ (or $d = d_G$, respectively).

It is well-known that for $p > n$, balls and other "nice" domains support (p,C)-Hölder imbeddings, with $C = C(n,p)$. More generally, it is shown in [BK2] that this imbedding also holds on all α-mCigar domains (as defined below) where $\alpha = (p-n)/(p-1)$; it is also shown there that the imbedding implies the α-mCigar condition if the domain satisfies a (strong) slice condition, as defined below.

1.6. Uniform domains and mean cigar domains.

Let $C \ge 1$ and let d be the Euclidean metric. We say that a domain G is a C*-uniform domain* if for every pair $x, y \in G$, there is a C*-uniform path*, i.e., a path $\gamma \in \Gamma_G(x,y)$ of length l and parametrized by arclength for which $l \le Cd(x,y)$, and $t \wedge (l-t) \le C\delta_G(\gamma(t))$. An *inner* C*-uniform domain* is defined similarly but with $d = d_G$. All uniform domains are inner uniform, while a slit disk is inner uniform but not uniform. For more on inner uniform domains, see [V].

Suppose that $0 \le \alpha \le 1 \le C$ and let $d : G \times G \to [0,\infty)$. We say that G is an $(\alpha,C;d)$*-mCigar domain* if for every pair $x, y \in G$, there is a $(\alpha,C;d)$*-mCigar path*, i.e., a path $\gamma \in \Gamma_G(x,y)$ such that

$$\begin{aligned} &\mathrm{len}_{\alpha,G}(\gamma) \le Cd(x,y)^\alpha \qquad 0 < \alpha \le 1, \\ &\mathrm{len}_{0,G}(\gamma) \le C \log[1 + d(x,y)/(\delta_G(x) \wedge \delta_G(y))], \qquad \alpha = 0. \end{aligned}$$

In particular, if d is the Euclidean metric, we simply say that G is an (α,C)*-mCigar domain*, while if $d = d_G$, we say that G is an *inner* (α,C)*-mCigar domain*. In practice we shall not use this terminology for $\alpha = 1$: we prefer to use the more common term C*-quasiconvex domain* rather than $(1,C)$-mCigar domain.

All uniform domains are α-mCigar domains for any choice of $\alpha \in [0,1]$. Gehring and Osgood [GO] showed that the classes of 0-mCigar domains and uniform domains coincide, and Väisälä [V, 2.33] showed that the classes of inner 0-mCigar and inner uniform domains coincide. Note that the classes of inner uniform and inner α-mCigar domains contain their Euclidean analogues (strictly, since a planar slit disk is in all of the inner classes but none of the Euclidean classes).

The role of the parameter α in the definition of an α-mCigar domain is rather subtle. First we note that the class of (inner) α'-mCigar domains includes the class of (inner) α-mCigar domains if and only if $\alpha \le \alpha'$. For the Euclidean case, see [L] and [BK2]; the inner case is similar. Lappalainen's examples [L, 6.7] of (non-uniform) domains that are β-mCigar but not α-mCigar, for each possible choice of $0 < \alpha < \beta \le 1$ makes use of a rather elaborate Cantor-type construction. It is intuitively clear that any such example must have a similar level of complexity. Thus domains that are easy to describe explicitly typically are either in all these classes or none of them. Among the examples of domains which are 0-mCigar, and so in all α-mCigar classes, are all bounded Lipschitz domains, as well as some domains with fractal boundary, such as the interior of a von Koch snowflake. In Section 4 we will give some further examples of mCigar domains.

We refer the reader to [BK2], [GM], and [L] for more information about α-mCigar domains, which elsewhere go under the aliases "weak cigar domains" and "Lip_α extension domains". We use the term "mean cigar" because these conditions imply the existence of a path γ that satisfies a type of cigar condition on average; see [BK2, Lemma 2.2] and also [BS, Lemma 4.3]. In this paper, we reserve the adjective "weak" for the slice conditions defined in the next section which, in particular, are satisfied by all planar simply-connected domains. By contrast mean cigar conditions rather strongly restrict the geometry of the domain: for instance, the proof of Proposition 4.6 will show that mCigar domains possess neither internal nor external cusps.

2. Slice domains

The conditions defined in Section 1 rather strongly restrict the geometry. For instance, among planar domains, inner uniform domains cannot have external cusps, while uniform and mCigar domains can have neither internal nor external cusps. By contrast, the slice conditions that we define in this section are all quite weak, at least in two dimensions: they are satisfied by any domain quasiconformally equivalent to a uniform domain and hence by all simply-connected planar domains.

2.1. Weak slice domains.

The basic Euclidean 0-wSlice condition defined below is essentially taken from [BO], where it is assumed uniformly for all x and a fixed y, but the $\alpha > 0$ case and non-Euclidean variants have not been considered before. We also prove some basic properties of these weak slice conditions in this subsection. The adjective "weak" refers to the fact that for all α, an α-wSlice condition is implied by the analogous "strong" slice condition which we define later.

Suppose $0 \le \alpha < 1 \le C$ and let d be a metric on G satisfying $d_{\mathbb{R}^n} \le d \le d_G$. Then G is an *$(\alpha, C; d)$-wSlice domain* if every pair $x, y \in G$ satisfies the following *$(\alpha, C; d)$-wSlice condition*: there exist a path $\gamma \in \Gamma_G(x, y)$, pairwise disjoint open subsets $\{S_i\}_{i=1}^m$ of G, $m \ge 0$, and numbers $d_i \in [\mathrm{dia}_d(S_i), \infty)$ such that:

$$\mathrm{len}(\lambda \cap S_i) \ge d_i/C, \qquad \lambda \in \Gamma_G(x, y),\ 1 \le i \le m; \tag{WS-1}$$

$$\mathrm{len}_{\alpha,G}(\gamma) \le C(\delta_G^\alpha(x) + \delta_G^\alpha(y) + \sum_{i=1}^m d_i^\alpha); \tag{WS-2}$$

$$(B(x, \delta_G(x)/C) \cup B(y, \delta_G(y)/C)) \cap S_i = \emptyset, \qquad 1 \le i \le m. \tag{WS-3}$$

If d is the Euclidean metric, we say that G is an *(α, C)-wSlice domain*, while if $d = d_G$, we say that G is an *inner (α, C)-wSlice domain*.

We will soon be looking carefully at some of the consequences of this condition and the inter-relationships between its three parts. We point out first of all that by (WS-1), each of the weak slices S_i must separate x from y in the domain G. The α-wSlice conditions are all (strictly) weaker than the slice conditions which were introduced by the first author and Koskela [BK2], as is shown in Lemma 2.8 below. This latter class is already quite vast since it includes (by Theorem 3.2 in [BK2]) all quasiconformal images of uniform domains and in particular, by the Riemann mapping theorem, all simply connected planar domains. In Theorem 3.1 of [BS], we go further to show that "uniform" can be replaced by "inner uniform" in the above result. There is, however, one significant difference between Slice and α-wSlice conditions: we shall see that every α-mCigar condition for a pair of points implies an α-wSlice condition for that pair, but we shall also see in the final section that, for every $\alpha > 0$, there exist non-Slice domains which are nevertheless α-mCigar (and so α-wSlice) domains.

Before going on, we now present two examples of planar domains D which are not α-wSlice. Let us fix $0 < \alpha < 1$ and write $p = (n-\alpha)/(1-\alpha) > n$. In general, it is a difficult task to show directly from the definition that a domain is not α-wSlice, so we shall use Theorem 0.1. Let $B = B(0,1)$ be the unit disk. For each positive integer n, consider the annulus

$$A_n = \{z \in B \,:\, 1/(n+1) < 1 - |z| < 1/n\}.$$

Inside A_n, we delete a finite set of points P_n so that $\delta_{B\setminus P_n}(z) < 2^{-n}$ for each z in A_n. Put $D = B \setminus \bigcup_{n=1}^{\infty} P_n$. Since isolated points are removable singularities of p-Hölder continuous functions and, for any $\alpha \in (0,1)$, B supports a p-Hölder imbedding, so must the domain D. If D were to be an α-wSlice domain then Theorem 0.1 would imply that D is also an α-mCigar domain. But this is obviously not the case since for example, $\lim_{t\to 1-} d_{\alpha,D}((0,0),(t,0)) = \infty$.

A similar example, which the reader may consider "less trivial", is produced by replacing every point $x_{n,i} \in P_n$ by a closed ball $\overline{B_{n,i}}$ so small that the concentric double dilates of these balls are pairwise disjoint; we again call the resulting domain D. Since uniform domains are $W^{1,p}$-extension domains [J, Theorem 1], we can take a function $f \in W^{1,p}(D)$, extend the functions $f|_{2B_{n,i}\setminus\overline{B_{n,i}}}$ to $\overline{B_{n,i}}$, and glue together these extensions to get an extension F of f with $\|F\|_{W^{1,p}(B)}$ comparable to $\|f\|_{W^{1,p}(D)}$. It follows from classical results that $F \in \mathrm{Lip}_{1-n/p}(B)$, and so $f \in \mathrm{Lip}_{1-n/p}(D)$. This argument together with Theorem 3.8 implies that D supports a p-Hölder imbedding. As before, however, D is not an α-mCigar, and hence not an α-wSlice, domain.

In the two examples just presented, the obstacles (i.e., removed points or balls) ensure that there are alternative $d_{\alpha,D}$-quasi-optimal paths between pairs of points which are not close in the Hausdorff (set) metric defined induced by the metric $d_{\alpha,D}$. This is the typical situation in which slice conditions fail. Another geometric configuration with this property is a "flat plate" in three (or more) dimensions, i.e. a box which is large in at least two dimensions, but small in at least one other dimension. Showing that a suitably constructed domain with many flat plates is not an α-wSlicedomain is, however, rather tricky; we refer the interested reader to the example after Open Problem B in [BS, Section 6].

For $\alpha = 0$, (WS-2) simply says that $\operatorname{len}_{0,G}(\gamma) \le C(2+m)$ and so if $\alpha = 0$, we can take $d_i = \operatorname{dia}_d(S_i)$ in the $(\alpha; d)$-wSlice condition. Although not obvious, we shall see below that this can also be done for $\alpha > 0$ (modulo a change in the value of C); however allowing inequality is sometimes convenient. For $\alpha = 0$, (WS-3) is an essential part of the definition (lest every domain be a $(0; d)$-wSlice domain), but when $\alpha > 0$ it can be dropped; see [BS, Theorem 4.12]. Obviously an inner α-wSlice condition implies an α-wSlice condition; the converse is false [BS, Section 5].

Modulo a change in the value of C by a factor at most 4, we may add the following condition to the definition of an $(\alpha, C; d)$-wSlice condition for x, y:

$$\operatorname{len}_{\alpha,G}(\gamma \cap S_i) \le C d_i^\alpha, \qquad 1 \le i \le m. \tag{WS-4}$$

To see this suppose that $x, y \in G$ satisfy an $(\alpha, C/4; d)$-wSlice condition for some $C \ge 4$, with slice data γ, $\{S_i', d_i'\}_{i=1}^{m'}$. We define new $(\alpha, C/2; d)$ slice data γ, $\{S_i, d_i\}_{i=1}^m$ satisfying (WS-4) as follows. First, we may assume that $2\operatorname{len}_{\alpha,G}(\gamma) \le C\sum_{i=1}^{m'}(d_i')^\alpha$, since otherwise we simply take $m = 0$. We discard S_i' if $\operatorname{len}_{\alpha,G}(\gamma \cap S_i') > C(d_i')^\alpha$, relabel the remaining ones as $\{S_i\}_{i=1}^m$, and relabel the numbers d_j' in the same fashion, so that $d_i = d_j'$ whenever $S_i = S_j'$. An easy calculation shows that $\sum_{i=1}^m d_i^\alpha \ge \sum_{i=1}^{m'}(d_i')^\alpha/2$, and so we are done.

We now prove a few lemmas concerning weak slice conditions.

Lemma 2.2. *Suppose that $c \in (0,1)$, that x, y are points in a bounded domain $G \subset \mathbb{R}^n$, and that $S \subset G \setminus \overline{B_x \cup B_y}$ is open, where $B_w \equiv B(w, c\delta_G(w))$, $w \in G$. Suppose further that every $\gamma \in \Gamma_G(x,y)$ intersects S. Then* $\operatorname{dia}(S) > 2c\delta_G(w)/(1+c)$ *for every $w \in S$.*

Proof. Suppose for the purpose of contradiction that the lemma is false. Choose $z \in \overline{S}$ such that $\delta \equiv \delta_G(z) = \max_{w\in\overline{S}} \delta_G(w)$. Thus $\operatorname{dia}(S) \le c'\delta$ and $S \subset B(z, c'\delta)$, where $c' \equiv 2c/(1+c) \in (0,1)$. Let us get a contradiction first under the additional assumption that $\overline{B_x} \subset B(z, c'\delta)$, which of course implies that $\overline{B_x} \subset B(z, r)$ for some $r < c'\delta$. Since $d(B(x,r), \partial G) \le d(B_x, \partial G)$, it follows that

$$(1-c')\,\delta < \delta - r \le (1-c)\,\delta_G(x),$$

and so $\delta < (1+c)\delta_G(x)$. But $z \notin B_x$ and so $c'\delta > r \ge 2c\delta_G(x)$, which in turn implies that $\delta \ge (1+c)\delta_G(x)$, giving the desired contradiction.

In view of the above argument, and a similar one with y replacing x, we may assume without loss of generality that there exist points $x' \in \overline{B_x} \setminus B(z, c'\delta)$ and $y' \in \overline{B_y} \setminus B(z, c'\delta)$. Let $\lambda \in \Gamma_G(x,y)$ be a path that has $[x \to x']$ as an initial segment and a reparametrized $[y' \to y]$ as a final segment. If λ intersects S, it must do so on the remaining middle segment, and it must pass through points x'', y'' of first and last contact with $\overline{B(z, c'\delta)}$. We replace the part of λ between x'' and y'' by a suitably parametrized arc on $\partial B(z, c'\delta)$ between x'' and y'' to get a path γ that avoids $B(z, c'\delta) \supset S$. This contradicts the hypotheses, so the lemma must be true. $\square$

According to the next lemma, (WS-1) and (WS-2) together imply that the slices for a pair of points $x, y \in G$ are "neither too large nor too thin" and cover at least some fixed fraction of the $d_{\alpha,G}$-length of any efficient path from x to y.

Lemma 2.3. *If the data γ, $\{S_i, d_i\}_{i=1}^m$ satisfy (WS-1) and (WS-3) for the pair $x, y \in G$, and $d_i > 0$, then $\operatorname{dia}(S_i) \geq 2\delta_G(z)/(C+1)$, for all $z \in S_i$ and $1 \leq i \leq m$. Furthermore, if $d_i \geq \operatorname{dia}(S_i)$ and $|x-y| \geq (\delta_G(x) + \delta_G(y))/2$, then there exists a constant $C' = C'(C, \alpha)$ such that*

$$\delta_G^\alpha(x) + \delta_G^\alpha(y) + \sum_{k=1}^m d_i^\alpha \leq C' \operatorname{len}_{\alpha,G}(\lambda), \qquad \lambda \in \Gamma_G(x,y). \tag{2.4}$$

Proof. The first statement follows from the previous lemma by letting c increase towards $1/C$. It is also easy to see that $\operatorname{len}_{\alpha,G}(\lambda \cap B(z, \delta_G(z)/C)) \gtrsim \delta_G^\alpha(z)$ for $z \in \{x, y\}$. If $d_i \geq \operatorname{dia}(S_i)$, then combining the first statement of the lemma with (WS-1), we see that $\operatorname{len}_{\alpha,G}(\lambda \cap S_i) \gtrsim d_i^\alpha$. By combining these estimates for disjoint pieces of λ, we deduce (2.4). □

Our next result carries two more lessons about slices. Ignoring a quantitatively controlled change in C, we may change "$d_i \geq \operatorname{dia}_d(S_i)$" to "$d_i = \operatorname{dia}_d(S_i)$" in the definition of an $(\alpha, C; d)$-wSlice condition, and we may assume that the d-diameter and Euclidean diameter of slices are comparable.

Lemma 2.5. *Suppose that $0 \leq \alpha < 1$ and that x, y are two points in a domain $G \subset \mathbb{R}^n$ that satisfy an $(\alpha, C; d)$-wSlice condition for some metric d, $d_{\mathbb{R}^n} \leq d \leq d_G$. Then there exist $(\alpha, 4C; d)$-wSlice data γ, $\{S_i, d_i\}_{i=1}^m$ for x, y such that $d_i \leq C' \operatorname{dia}(S_i)$ for some $C' = C'(C, \alpha)$.*

Proof. It follows from the discussion after (WS-4) that there exist $(\alpha, 2C; d)$-wSlice data γ, $\{S_i, d_i\}_{i=1}^m$ for x, y that satisfy (WS-4) (with C replaced by $4C$). Any such set of data will have the property that we seek. Consider the case $\alpha = 0$. Let $\delta_i = \sup_{\gamma \cap S_i} \delta_G(z)$. Using (WS-4), we obtain $\operatorname{len}(\gamma \cap S_i)\delta_i^{-1} \leq \operatorname{len}_{0,G}(\gamma \cap S_i) \lesssim 1$. But by (WS-1) we have $d_i \lesssim \operatorname{len}(\gamma \cap S_i)$ and so combining these two estimates gives the desired estimate $d_i \lesssim \operatorname{dia}(S_i)$.

The case $\alpha > 0$ is a little trickier. Again we let $\delta_i = \sup_{\gamma \cap S_i} \delta_G(z)$. For each $k \in \mathbb{N}$, let $l_{i,k}$ be the total length of that part of $\gamma \cap S_i$ on which the distance to ∂G lies in the range $(2^{-k}\delta_i, 2^{-k+1}\delta_i]$. Lemma 2.3 and the fact that $d \geq d_{\mathbb{R}^n}$ imply that $\delta_i \lesssim \operatorname{dia}(S_i) \leq d_i$. By (WS-4), we have

$$\sum_{k=1}^\infty l_{i,k} \delta_i^{\alpha-1} 2^{k(1-\alpha)} \lesssim \operatorname{len}_{\alpha,G}(\gamma \cap S_i) \lesssim d_i^\alpha,$$

and so

$$l_{i,k} \lesssim \left(\frac{d_i}{\delta_i}\right)^{\alpha-1} 2^{-k(1-\alpha)} d_i \lesssim 2^{-k(1-\alpha)} d_i. \tag{2.6}$$

But by (WS-1), $\sum_{k=1}^\infty l_{i,k} = \operatorname{len}(\gamma \cap S_i) \gtrsim d_i$. Combining this with (2.6), it follows that $l_{i,k} \gtrsim d_i$ for some $k \leq k_0 = k_0(C, \alpha)$. But then by the first half of (2.6), we see that $d_i^{1-\alpha} \lesssim \delta_i^{1-\alpha}$, and so $d_i \lesssim \delta_i \lesssim \operatorname{dia}(S_i)$. □

2.7. "Strong" slice domains.

Suppose $C \geq 1$ and let d be a metric on G satisfying $d_{\mathbb{R}^n} \leq d \leq d_G$. Then G is a $(C; d)$-*Slice domain* if every pair $x, y \in G$ satisfies the following $(C; d)$-*Slice condition*: there exist a path $\gamma \in \Gamma_G(x, y)$ and pairwise disjoint open subsets $\{S_i\}_{i=1}^m$ of G, with $d_i \equiv \mathrm{dia}_d(S_i) < \infty$, such that:

(i) $x \in S_0$, $y \in S_j$, and x and y are in different components of $G \setminus \overline{S_i}$, for all $0 < i < j$.
(ii) $\mathrm{len}(\lambda \cap S_i) \geq d_i/C$, for all $0 < i < j$ and $\lambda \in \Gamma_G(x, y)$.
(iii) For all $t \in [0, 1]$, we have $B\left(\gamma(t), C^{-1}\delta_G(\gamma(t))\right) \subset \bigcup_{i=0}^j \overline{S_i}$. Also, for all $0 \leq i \leq j$, there exists $x_i \in \gamma_i$, such that $x_0 = x$, $x_j = y$, and $B\left(x_i, C^{-1}\delta_G(x_i)\right) \subset S_i$.
(iv) For all $0 \leq i \leq j$ and $z \in \gamma_i \equiv \gamma([0, 1]) \cap S_i$, we have $d_i \leq C\delta_G(z)$.

If d is the Euclidean metric, we say that G is a C-*Slice domain*, while if $d = d_G$, we say that G is an *inner* C-*Slice domain*. The (Euclidean) Slice condition was first defined in [BK2], where it is assumed uniformly for all x and a fixed y.

We now show that Slice domains are α-wSlice domains for every α.

Lemma 2.8. *If the pair $x, y \in G$ satisfies the $(C; d)$-Slice condition for some metric d satisfying $d_{\mathbb{R}^n} \leq d \leq d_G$, then for each $\alpha \in [0, 1)$, x, y satisfies the $(\alpha, C'; d)$-wSlice condition for some $C' = C'(C, \alpha, n)$.*

Proof. If γ, $\{S_i\}_{i=0}^j$ are the $(C; d)$-Slice data for the pair x, y, then the required $(\alpha, C'; d)$-wSlice data are $\tilde{\gamma}$, $\{S_i, d_i\}_{i=1}^m$, where $m = j - 1$, $d_i = \mathrm{dia}_d(S_i)$, and $\tilde{\gamma}$ will be defined. Slice properties (ii), (iii) immediately imply (WS-1), (WS-3), so it suffices to verify (WS-2).

Properties (iii) and (iv) imply that $d_i/\delta_G(x_i) \in (1/C, C]$, and that if we write $B_z \equiv B(z, d_i/2C^2)$, then $2B_z \subset G$ for every $z \in \gamma_i$, $0 \leq i \leq j$. For fixed i, any pairwise disjoint subset of $\{(1/3)B_z : z \in \gamma_i\}$ has finite cardinality, with a bound dependent only on C and n. Taking a maximal pairwise disjoint subcollection, it is clear that the associated collection of dilated balls of the form B_z covers $\overline{\gamma_i}$. We relabel these latter balls $\mathcal{B}_i \equiv \{B_i^j\}_{j=1}^{k(i)}$, where $k(i) \leq k_0 = k_0(C, n) < \infty$.

We now replace the path γ with a polygonal path by the following finite incremental polygonalization procedure involving a partition $\{0 = s_0 < s_1 < \cdots < s_M = 1\}$ of $[0, 1]$ which we shall construct, and paths γ^k which all have the same value at each s_i and are polygonal as far as $t = s_k$. First let $s_0 = 0$, $i_0 = 0$, let j_0 be such that $x \in B_{i_0}^{j_0}$, and let s_1 be the largest value of $t \in [0, 1]$ such that $\gamma(t) \in \overline{B_{i_0}^{j_0}}$. We define a new path γ^1 to be the same as γ except that we replace the path segment $\gamma|_{[s_0, s_1]}$ by a line segment from $\gamma(s_0)$ to $\gamma(s_1)$.

For the general inductive step, we assume that we have already defined s_m and γ^m for all $0 \leq m \leq k$. If $s_k = 1$, the procedure is declared to be complete. Otherwise, let i_k, j_k be such that $\gamma^k(s_k) \in B_{i_k}^{j_k}$, and let s_{k+1} be the largest value of $t \in [0, 1]$ such that $\gamma(t) \in \overline{B_{i_k}^{j_k}}$. Predictably, we now replace the path segment $\gamma|_{[s_k, s_{k+1}]}$ by $[\gamma(s_k) \to \gamma(s_{k+1})]$.

The balls $B_{i_k}^{j_k}$ are all distinct, so this process must terminate; let $\tilde{\gamma}$ be the final, fully polygonalized, path. A straightforward calculation gives $\mathrm{len}_{\alpha,G}(\tilde{\gamma}|_{[s_k, s_{k+1}]}) \leq$

$Cd_{i_k}^{\alpha}$, where $C' = C'(C, \alpha, n)$, and so $\mathrm{len}_{\alpha,G}(\tilde{\gamma}) \le k_0 C' \sum_{i=0}^{m+1} d_i^{\alpha}$, where $m \equiv j-1$. The lemma follows since $d_0 \lesssim \delta_G(y)$ and $d_{m+1} \lesssim \delta_G(x)$. $\square$

Cigar conditions always imply the corresponding slice conditions, as implied by the following slightly weakened form of [BS, Lemma 3.4].

Lemma 2.9. *Suppose that $0 \le \alpha < 1$ and that $G \subsetneq \mathbb{R}^n$. If there is an inner (α, C)-mCigar path for the points $x, y \in G$, then the pair x, y satisfies an inner (α, C')-wSlice condition for some $C' = C'(C, \alpha, n)$. If $\alpha = 0$, x, y also satisfies an inner C''-Slice condition for some $C'' = C''(C, n)$.*

3. Hölder-type imbedding theorems

Generalizing Theorem 0.1, we shall prove the following result.

Theorem 3.1. *Let $0 < \alpha < 1 < C$, $G \subsetneq \mathbb{R}^n$, and suppose that d is a metric on G satisfying $d_{\mathbb{R}^n} \le d \le d_G$. Then G is an $(\alpha, C_1; d)$-mCigar domain if and only if it is an $(\alpha, C_2; d)$-wSlice domain which supports a $(p, C_3; d)$-Hölder imbedding for $p = (n-\alpha)/(1-\alpha) > n$. The constants C_1, C_2, C_3 depend only on each other, and on α and n.*

Actually, the exact type of slice condition used in this theorem does not matter, in the sense that an $(\alpha, C_1; d)$-mCigar domain satisfies the strongest condition of this type (i.e., it is an inner (α, C_2)-wSlice domain), while if G satisfies the weakest condition of this type (i.e., it is an (α, C_2)-wSlice domain), then that together with an $L^{1,p}(G, d) \hookrightarrow \mathrm{Lip}_{1-n/p}(G, d)$ imbedding implies that G is an $(\alpha, C_1; d)$-mCigar domain. This stronger version of Theorem 3.1 follows by combining Lemma 2.9 with the following theorem.

Theorem 3.2. *Let $G \subsetneq \mathbb{R}^n$, $0 < \alpha < 1$, and $p = (n-\alpha)/(1-\alpha) > n$. Then there exists a constant $C = C(n, \alpha)$ such that for all $u \in L^{1,p}(G)$ and all $x, y \in G$,*

$$|u(x) - u(y)| \le C[d_{\alpha,G}(x, y) + |x-y|^{\alpha}]^{(p-1)/p} \|\nabla u\|_{L^p(G)}, \tag{3.3}$$

Conversely, if $x, y \in G$ satisfy an (α, C_0)-wSlice condition then this inequality can be reversed for some $u \in L^{1,p}(G)$ (dependent on x, y), and $C = C(n, \alpha, C_0) > 0$.

Proof. Throughout this proof, we write $B_z = B(z, \delta_G(z)/2)$, for all $z \in G$. We first prove (3.3); this proof is similar to the proof of sufficiency for Theorem 4.1(iii) in [BK2], but we include it for completeness.

Note that $\alpha = (p-n)/(p-1)$. If $y \in 2B_x$, then (3.3) follows from the classical inequality for balls $B \subset \mathbb{R}^n$:

$$|u(x) - u(y)| \le C|x-y|^{(p-n)/p} \|\nabla u\|_{L^p(B)}, \qquad u \in L^{1,p}(B),\ x, y \in B, \tag{3.4}$$

where $C = C(n, p)$. For a proof of this, see [Z, 2.4.4].

Suppose therefore that $y \notin 2B_x$. Let $\gamma \in \Gamma_G(x, y)$ be such that $\mathrm{len}_{\alpha,G}(\gamma) \le 2d_{\alpha,G}(x, y)$. We cover γ by the balls $B_{\gamma(t)}$, $0 \le t \le 1$. Note that the length of $\gamma \cap B_{\gamma(t)}$ is at least $\delta_G(\gamma(t))/2$, that all points in $B_{\gamma(t)}$ are approximately the same distance from ∂G. By compactness and the Besicovitch Covering Lemma [S2, I.8.17], we can extract a subcollection $\mathcal{B} = \{B^i\}_{i=0}^j$ of $\{B_{\gamma(t)}\}$ such that $\mathcal{B}$ still covers γ but no point in G lies in more than $C = C(n)$ of the balls of $\mathcal{B}$. We arrange the indices so that there are points $\{x_i\}_{i=0}^j$ with $x_0 = x$, $x_j = y$, and $x_i \in B^{i-1} \cap B^i$ for $i = 1, \dots, j$. By the triangle inequality, (3.4), and Hölder's inequality, we get

$$\begin{aligned}
|u(x) - u(y)| &\lesssim \sum_{i=1}^{j} |x_i - x_{i-1}|^{(p-n)/p} \|\nabla u\|_{L^p(B^i)} \\
&\le \left(\sum_{i=1}^{j} |x_i - x_{i-1}|^{\frac{p-n}{p-1}} \right)^{(p-1)/p} \left(\sum_{i=1}^{j} \|\nabla u\|^p_{L^p(B^i)} \right)^{1/p} \\
&\lesssim \left(\sum_{i=1}^{j} \int_{\gamma \cap B^i} \delta_G(z)^{\frac{1-n}{p-1}} \right)^{(p-1)/p} \|\nabla u\|_{L^p(G)} \\
&\lesssim (\operatorname{len}_{\alpha,G} \gamma)^{(p-1)/p} \|\nabla u\|_{L^p(G)}.
\end{aligned}$$

For the converse, we assume without loss of generality that $\delta_G(y) \le \delta_G(x)$ and $x \ne y$. Consider first the case $y \in \overline{B_x}$; we shall not need the slice condition in this case. Then $d_{\alpha,G}(x,y) \lesssim |x-y|^\alpha$, so we simply need a function which reverses the inequality in (3.4). As the reader can verify, one such function is given by

$$u(z) = [|x-y| \vee 2(|x-y| \wedge |x-z|)]^{1-n/p}, \qquad z \in G.$$

We next consider the case $y \notin \overline{B_x}$. Letting $\mu : [0,l] \to G$ be a path in $\Gamma_G(x,y)$ such that $\operatorname{len}_{\alpha,G}(\mu) \le 2d_{\alpha,G}(x,y)$, we see that

$$2d_{\alpha,G}(x,y) \ge \operatorname{len}(\mu) \cdot M^{\alpha-1}, \quad \text{where } M = \max_{z \in \mu([0,l])} \delta_G(z).$$

But $M \le \operatorname{len}(\mu) + \delta_G(x) \le 3\operatorname{len}(\mu)$, and so $d_{a,G}(x,y) \gtrsim \operatorname{len}(\mu)^\alpha \ge |x-y|^\alpha$. Thus it suffices to find $\gamma \in \Gamma_G(x,y)$ and a function u such that $\|\nabla u\|_{L^p(G)} = 1$ and $|u(x) - u(y)| \gtrsim \operatorname{len}_{\alpha,G}(\gamma)^{(p-1)/p}$.

Let γ, $\{S_i, d_i\}_{i=1}^m$ be (α, C_0)-wSlice data for x, y. If $\operatorname{len}_{\alpha,G}(\gamma) \le 3C_0\delta_G^\alpha(x)$, then the example given in the previous case for a point $y \in \partial B_x$ works for the current y also. We may therefore assume that $\operatorname{len}_{\alpha,G}(\gamma) \ge 3C_0\delta_G^\alpha(x)$. Together with (WS-2), this gives $\operatorname{len}_{\alpha,G}(\gamma) \le 3C_0 \sum_{i=1}^m d_i^\alpha$.

For $1 \le i \le m$, let

$$u_i(z) = \frac{c_i}{d_i} \left[\inf_{\lambda \in \mathcal{F}_{z,x}} \operatorname{len}(\lambda \cap S_i) \right], \qquad z \in G,$$

where $\mathcal{F}_{z,x}$ is the set of all rectifiable curves in G containing both z and x, and c_i is a positive constant to be specified later. Let $g_i = \|\nabla u_i\|_{L^p(G)}$. Since ∇u_i is supported on S_i, and $|\nabla u_i| \le c_i/d_i$, we have $g_i \le (c_i/d_i)|S_i|^{1/p} \le c_i d_i^{n/p-1}$. Thus, using (WS-1), it follows that

$$u_i(y) - u_i(x) \gtrsim c_i \ge d_i^{1-n/p} g_i.$$

Defining $u = \sum_{i=1}^m u_i$, it follows that $u(y) - u(x) \gtrsim \sum_{i=1}^m d_i^{1-n/p} g_i$. Since we have not yet specified c_i, we are free to choose g_i arbitrarily. Let $g_i = c d_i^{(1-n/p)/(p-1)}$ where c is chosen so that $\sum_{i=1}^m g_i^p = 1$. It follows that $\|\nabla u\|_{L^p(G)} = 1$ and that

$$u(y) - u(x) \gtrsim \left(\sum_{i=1}^m d_i^{(p-n)/(p-1)} \right)^{(p-1)/p} \gtrsim (\mathrm{len}_{\alpha,G}(\gamma))^{(p-1)/p}. \quad \square$$

Theorem 3.2 also implies Hölder imbedding results for functions d much more general than the metrics d considered in Theorem 3.1.

Theorem 3.5. *Suppose that $0 < \alpha < 1$, C, and C' are positive constants, that $G \subset \mathbb{R}^n$ is a bounded (α, C)-wSlice domain, and that $d : G \times G \to [0, \infty)$ is such that $d \geq C' d_{\mathbb{R}^n}$. Then G is an $(\alpha, C_1; d)$-mCigar domain if and only if it supports an $(p, C_2; d)$-Hölder imbedding for $p = (n-\alpha)/(1-\alpha) > n$. The constants C_1 and C_2 depend only on each other and on p, n, C, and C'.*

For example, this last theorem gives Hölder imbedding results with exponents less than the p-Hölder exponent $1 - n/p$ if we take $d(x, y) = |x - y|^t$ for some $0 < t < 1$; cf. [BK2, Theorem 5.2].

We can also prove imbedding results where boundedness replaces Hölder continuity. For example, we have the following analogue of Theorem 3.1.

Theorem 3.6. *Suppose that $0 < \alpha < 1$ and that $G \subset \mathbb{R}^n$ is a bounded domain with $x_0 \in G$. Then the $d_{\alpha,G}$-diameter of G is at most C_1 if and only if the pair x, x_0 satisfies an (α, C_2)-wSlice condition for all $x \in G$, and G supports the L^∞ imbedding*

$$|u(x) - u(x_0)| \leq C_3 \|\nabla u\|_{L^p(G)}, \qquad u \in L^{1,p}(G). \tag{3.7}$$

The constants C_1, C_2, C_3 depend only on each other, and on x_0, α, and n.

Proof. The $d_{\alpha,G}$-boundedness of G implies an α-wSlice condition for x, x_0 with zero slices and, in view of Theorem 3.2, it also implies (3.7). Theorem 3.2 also implies the converse direction. $\square$

We next wish to discuss the imbedding $W^{1,p}(G) \hookrightarrow \mathrm{Lip}_{1-n/p}(G, d)$, $p > n$. When $d = d_{\mathbb{R}^n}$ and G is bounded, this is equivalent to the imbedding $L^{1,p}(G) \hookrightarrow \mathrm{Lip}_{1-n/p,\epsilon}(G, d)$; see [KR]. We shall show that this equivalence and others extend to imbeddings defined in terms of any of a large class of functions d. The proof of equivalence, an adaptation of the methods of Koskela and Reitich [KR] for the Euclidean metric, is independent of a relationship between the Sobolev and Lipschitz exponents; however, since d is allowed to be quite general, this decoupling is only a convenience and not a generalization. With these equivalences in hand, we can use our earlier methods to find conditions for a domain to support these imbeddings.

We define the variational p-capacity $\mathrm{cap}_p(E, F; G)$, where E, F are compact subsets of $\overline{G}$. First let $L(E, F; G)$ denote the class of all functions $u \in L^{1,p}(G)$ that are continuous on $G \cup E \cup F$ and equal C_0 on E and C_1 on F, for some numbers C_0, C_1 satisfying $|C_1 - C_0| = 1$. Then

$$\operatorname{cap}_p(E, F; G) = \inf_{u \in L(E,F;G)} \int_G |\nabla u|^p.$$

We get an equivalent definition of $\operatorname{cap}_p(E, F; G)$ if we replace $L(E, F; G)$ by its subset $L_0(E, F; G)$ consisting of those functions whose values lie between $C_0 = 0$ and $C_1 = 1$; see [Mz, 4.1.1]. We abbreviate singleton sets $\{x\}$ to x when dealing with capacities.

Theorem 3.8. *Let $G \subset \mathbb{R}^n$ be a domain and let $s \in (0,1)$, $p \in (n, \infty)$. Suppose that $d : G \times G \to [0, \infty)$ satisfies $\psi(|x - y|) \le d(x, y) \le \phi(|x - y|)$, where $\psi, \phi : (0, \infty) \to (0, \infty)$ are non-decreasing functions and $\lim_{t \to 0^+} \phi(t) = 0$. Then the following are equivalent:*

(i) $W^{1,p}(G) \hookrightarrow C^{0,s}(G, d)$;
(ii) $W^{1,p}(G) \hookrightarrow \operatorname{Lip}_s(G, d)$;
(iii) $W^{1,p}(G) \hookrightarrow \operatorname{Lip}_{s,\epsilon_1}(G, d)$ *for some* $\epsilon_1 > 0$;
(iv) $L^{1,p}(G) \hookrightarrow \operatorname{Lip}_{s,\epsilon_2}(G, d)$ *for some* $\epsilon_2 > 0$;
(v) *There exists some* $\epsilon_3 > 0$ *such that* $\operatorname{cap}_p(x, y; G) \ge C d(x, y)^{-ps}$ *whenever* $x, y \in G$, $|x - y| \le \epsilon_3$.

If G is bounded, (i)–(v) above are also equivalent to

(vi) $L^{1,p}(G) \hookrightarrow \operatorname{Lip}_s(G, d)$.

Furthermore C, the various imbedding constants, and the numbers ϵ_i depend only on each other, ϕ, ψ, s, p, n, and (if (vi) is the implied condition) $\operatorname{dia}(G)$.

We shall need three lemmas, the first of which is due to Maz'ya [Mz, 5.1.1].

Lemma 3.9. *Suppose $p \in (n, \infty)$ and $G \subset \mathbb{R}^n$ is a domain. Then $W^{1,p}(G) \hookrightarrow L^\infty(G)$ if and only if there exist numbers $r, C > 0$ such that $\operatorname{cap}_p(x, \overline{G} \setminus B(x, r); G) \ge C$ for all $x \in G$. Moreover the imbedding norm and the constants r, C depend only on each other, p, and n.*

Lemma 3.10. *Suppose $p \in (n, \infty)$, $0 < \epsilon \le \infty$, that $G \subset \mathbb{R}^n$ is a domain, and that $d : G \times G \to [0, \infty)$ is positive off the diagonal. Then $L^{1,p}(G) \hookrightarrow \operatorname{Lip}_{s,\epsilon}(G, d)$ with imbedding constant C if and only if $\operatorname{cap}_p(x, y; G) \ge C^{-p} d(x, y)^{-ps}$ whenever $x, y \in G$, $|x - y| \le \epsilon$.*

Proof. The fact that the capacity condition follows from the imbedding is obvious. For the converse, assume that $u \in L^{1,p}(G)$ with $u(x) \ne u(y)$ for some $x, y \in G$, $|x - y| < \epsilon$. Applying the capacity condition to the function $z \mapsto u(z)/|u(x) - u(y)|$, the imbedding follows. $\square$

Lemma 3.11. *Let G, s, p, and ϕ be as in Theorem 3.8, and let $d : G \times G \to [0, \infty)$ be positive off the diagonal and satisfy $d(x, y) \le \phi(|x - y|)$. If for some $\epsilon > 0$, $W^{1,p}(G) \hookrightarrow \operatorname{Lip}_{s,\epsilon}(G, d)$ with imbedding constant C then for some $\epsilon' > 0$,*

$L^{1,p}(G) \hookrightarrow \mathrm{Lip}_{s,\epsilon'}(G,d)$ with imbedding constant $2C$. The number ϵ' depends only on $C\phi^s$, ϵ, p, and n.

Proof. Let $x, y \in G$ satisfy $|x-y| \le \epsilon$, and fix $\eta \in C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\eta \subset B(0,\epsilon)$, $\eta(0) = 1$, $|\nabla\eta| \le 2/\epsilon$, and $0 \le \eta \le 1$. Taking $u \in L_0(x,y;G)$, we see that $v(z) = u(z)\eta(z-y)$ defines a function in $W^{1,p}(G)$ and that

$$\begin{aligned} 1 = v(y) - v(x) &\le C^p d(x,y)^{ps} \int_G |v|^p + |\nabla v|^p \\ &\le C^p d(x,y)^{ps} \int_G \left[\chi_{B(y,\epsilon)} + 2^{p-1}\left(|\nabla\eta|^p + |\nabla u|^p\right)\right] \\ &\le C^p d(x,y)^{ps} \left[C_0 + 2^{p-1}\int_G |\nabla u|^p\right], \end{aligned}$$

where $C_0 = C_0(\epsilon, p, n)$. Thus

$$C^{-p} d(x,y)^{-ps} - C_0 \le 2^{p-1} \operatorname{cap}_p(x,y;G).$$

Choosing $\epsilon' > 0$ so small that $[C\phi(\epsilon')^s]^{-p} \ge 2C_0$, we see that if $x, y \in G$, $|x-y| \le \epsilon'$, then $\operatorname{cap}_p(x,y;G) \ge (2Cd(x,y)^s)^{-p}$. Hence Lemma 3.10 yields the claim. □

Proof of Theorem 3.8. It follows immediately from the definitions that (i) implies (ii), and (ii) implies (iii). By Lemma 3.11, (iii) implies (iv), and by Lemma 3.10, (iv) implies (v). We next show that (v) implies (i). First it follows from Lemma 3.9 that $W^{1,p}(G) \hookrightarrow L^\infty(G)$, so it suffices to show that $W^{1,p}(G) \hookrightarrow \mathrm{Lip}_s(G)$. By Lemma 3.10, we have $W^{1,p}(G) \hookrightarrow \mathrm{Lip}_{s,\epsilon_3}(G,d)$. Let $u \in W^{1,p}(G)$, and $x, y \in G$ with $|x-y| \ge \epsilon_3$. Since $W^{1,p}(G) \hookrightarrow L^\infty(G)$, we have

$$|u(x) - u(y)| \le 2\|u\|_{L^\infty(G)} \lesssim \|u\|_{W^{1,p}(G)} \lesssim \psi(\epsilon_3)^{-s} d(x,y)^s \|u\|_{W^{1,p}(G)}.$$

and (i) follows since $\psi(\epsilon_3)^{-s} \lesssim 1$.

We have shown that (i)–(v) are equivalent. It is clear that (vi) implies (iv). To finish the theorem, we prove (vi) under the assumptions that $\mathrm{dia}(G) \equiv \Delta < \infty$ and that (iv) holds with imbedding constant C. By a standard covering lemma (see [S1, I.6, I.7]) it follows that G can be covered by a countable collection of balls $\{B_i\}_{i=1}^N$ of radius ϵ_2 such that the smaller balls $\{(1/5)B_i\}_{i=1}^N$ are pairwise disjoint and the centers of these balls lie in G. Since these smaller balls are disjoint and all lie in an ϵ_2-neighbourhood of G, we must have $N \le (5(\Delta + \epsilon_2)/\epsilon_2)^n$. By chaining the triangle inequality, we see that $|u(x) - u(y)| \le CN\phi(\epsilon_2)^s \|\nabla u\|_{L^p(G)}$, whenever $x, y \in G$. This gives the desired imbedding inequality whenever $|x-y| \ge \epsilon_2$. Since we already know that the inequality holds when $|x-y| < \epsilon_2$, we are done. □

It is easy to apply Theorem 3.8 to get versions of our earlier imbedding theorems for the imbedding $W^{1,p}(G) \hookrightarrow \mathrm{Lip}_{1-n/p}(G,d)$. For instance, we shall state without proof a $W^{1,p}(G)$ version of Theorem 3.5 after some preliminary definitions.

Let $G \subsetneq \mathbb{R}^n$. We say that G is (α, C, ϵ)-wSlice domain, or an (inner) $(\alpha, C_1, \epsilon; d)$-mCigar domain, if every pair of points $x, y \in G$ with $|x-y| \le \epsilon$ satisfies an (α, C)-wSlice condition, or an (inner) $(\alpha, C; d)$-mCigar condition, respectively. If we do not care about the value of ϵ, we simply refer to *local (α, C)-wSlice* and *local (inner) $(\alpha, C; d)$-mCigar domains*; similarly, we can define *local (inner) uniform domains.* It is easy to see that a local (inner) uniform domain is a local (inner) α-mCigar domain for every $\alpha \in (0,1)$.

Theorem 3.12. *Suppose that $0 < \alpha < 1$, $p = (n-\alpha)/(1-\alpha) > n$, ϵ, C, and C' are positive constants, that $G \subset \mathbb{R}^n$ is a bounded (α, C, ϵ)-wSlice domain, and that $d : G \times G \to [0,\infty)$ is such that $C'|x-y| \le d(x,y) \le \phi(|x-y|)$ for some non-decreasing function $\phi : (0,\infty) \to (0,\infty)$ satisfying $\lim_{t\to 0^+} \phi(t) = 0$. Then G is an $(\alpha, C_1, \epsilon_1; d)$-mCigar domain for some $\epsilon > 0$ if and only if $W^{1,p}(G) \hookrightarrow \mathrm{Lip}_{1-n/p}(G,d)$ with imbedding constant C_2. The constants C_1, C_2, ϵ_1 depend only on each other and on C, C', ϕ, α, ϵ, p, and n.*

4. Imbedding examples and non-examples

We first apply the theorems in the last section to some specific domains. For notational convenience, we deal only with planar regions, but these examples can readily be generalized to higher dimensions; we denote the dimension in imbeddings by "n" as usual, rather than "2" to emphasise this. In all examples, we assume that $p \in (2,\infty)$ and look at imbeddings of the form $X \subset \mathrm{Lip}_{1-n/p}(G,d)$ for various functions d, where X is either $L^{1,p}(G)$ or $W^{1,p}(G)$. Every domain we consider is simply-connected and thus an α-wSlice domain for each $\alpha \in [0,1)$ by the Riemann mapping theorem together with Lemmas 2.8 and 2.9.

Example 4.1. For each $k \in \mathbb{N}$, let U_k be the roughly U-shaped region defined by

$$U_k \;=\; (0,1)\times[0,2) \;\cup\; (0,4)\times(1,2) \;\cup\; (3,4)\times(2^{-k},2)$$

and let L_k be the affine map defined by $L_k(z) = 2^{-k-3}z + (2^{-k},1)$. Next let $H_k = L_k(U_k)$ and $G = (0,1)^2 \cup (\bigcup_{k=1}^\infty H_k)$, so that G is a square with a sequence of almost-closed hooks attached. It is easy to verify that G is an inner uniform domain and hence an inner α-mCigar domain for every $\alpha \in [0,1)$. Since $|x-y| \le d_G(x,y) \le 2|x-y|^{1/2}$, it follows from Theorem 3.5 that $L^{1,p}(G) \hookrightarrow \mathrm{Lip}_{1-n/p}(G,d_G)$.

If $\alpha \in (0,1)$, then $d_{\alpha,G}(x,y) \le C_\alpha |x-y|^{\alpha/2}$, with approximate equality for certain pairs of points x, y with $|x-y|$ is arbitrarily small (to see this, pick $x \in H_k$ and $y \in (0,1)^2$ to be points near the boundary of G on either side of the narrow gap between H_k and $(0,1)^2$). Thus we have $L^{1,p}(G) \hookrightarrow \mathrm{Lip}_{1-n/p}(G, d^t_{\mathbb{R}^2})$ only for $t \le 1/2$; this is weaker than the previous inner metric imbedding since $d_G \le 2d^{1/2}_{\mathbb{R}^2}$. By Theorem 3.8, we cannot increase t even if we replace $L^{1,p}(G)$ by $W^{1,p}(G)$.

Example 4.2. Let U_k be as in Example 4.1, but now let $L_k(z) = 4^k z + (4^{k+3},0)$, $H_k = L_k(U_k)$, and $G = [\mathbb{R}\times(-\infty,0)] \cup (\bigcup_{k=1}^\infty H_k)$. As before, G is inner uniform and $L^{1,p}(G) \hookrightarrow \mathrm{Lip}_{1-n/p}(G,d_G)$. Writing $d_t(x,y) = |x-y| \vee |x-y|^t$, we have $L^{1,p}(G) \hookrightarrow \mathrm{Lip}_{1-n/p}(G,d_t)$ only when $t \ge 2$; note that d_2 is not even a metric. This imbedding is again weaker than the inner metric imbedding. By contrast with Example 4.1, replacing $L^{1,p}(G)$ by $W^{1,p}(G)$ makes quite a difference. In fact, G is a local uniform domain (with $\epsilon = 1$), and so $W^{1,p}(G) \hookrightarrow \mathrm{Lip}_{1-n/p}(G,d_{\mathbb{R}^2})$.

Example 4.3. Let $G = \mathbb{R}^2 \setminus [0,\infty)\times[-1,1]$. Then G is both a local uniform and an inner uniform domain, so $W^{1,p}(G) \hookrightarrow \mathrm{Lip}_{1-n/p}(G,d_{\mathbb{R}^2})$ and $L^{1,p}(G) \hookrightarrow \mathrm{Lip}_{1-n/p}(G,d_G)$. However $L^{1,p}(G) \not\hookrightarrow \mathrm{Lip}_{1-n/p}(G,d_{\mathbb{R}^2})$ since G is not an α-mCigar domain for any $\alpha \in [0,1)$.

The examples so far all support the imbedding $L^{1,p}(G) \hookrightarrow \mathrm{Lip}_{1-n/p}(G,d_G)$. To get examples for which this imbedding fails, we simply need some form of a cusp or long corridor in our domain.

Example 4.4. Let $G = \{(x_1, x_2) \in \mathbb{R}^2 \,:\, 0 < x_1 < \infty,\ |x_2| < x_1^{1/2}\}$. G is convex, so $d_G = d_{\mathbb{R}^2}$. Let $\alpha = (p-n)/(p-1)$, and for $t > 1$, let $d_t = d_G \vee d_G^t$. A little calculation shows that the minimal exponent for which we have $d_{\alpha,G} \le C d_t^\alpha$ is $t = (1+\alpha)/2\alpha$; note that the minimum is achieved for a pair of points $z = (1,0)$, $w = (w_1, 0)$, with $w_1 \to \infty$. Thus by Theorem 3.5, $L^{1,p}(G) \hookrightarrow \mathrm{Lip}_{1-n/p}(G, d_{(1+\alpha)/2\alpha})$, but in light of Theorem 0.1, $L^{1,p}(G) \not\hookrightarrow \mathrm{Lip}_{1-n/p}(G, d_G)$. However, G is a local uniform domain, so Theorem 3.12 implies that $W^{1,p}(G) \hookrightarrow \mathrm{Lip}_{1-n/p}(G, d_G)$.

Our next proposition provides specific settings in which the strong slice condition is much more geometrically restrictive than any wSlice condition.

Proposition 4.5. *Fix numbers $\alpha, c \in (0,1)$. Let $t_j \in (0, 1/2)$ with $t_{j+1} < ct_j$ for each $j \in \mathbb{N}$. Let T be the planar triangle $\{(x,y) : 0 < x < 1, |y| < x\}$ and let $F = \bigcup_{j=1}^\infty F_j$, where F_j consist of the n_j points which divide the line segment $\{(t_j, y) : |y| < t_j\}$ into $n_j + 1$ equal subsegments. Then $G \equiv T \setminus F$ is a quasiconvex (inner) α-mCigar domain, and hence an (inner) α-wSlice domain, but is an (inner) Slice domain if and only if the sequence (n_j) is bounded.*

Proof of Proposition 4.5. It is clear that G is quasiconvex, so every inner slice condition is equivalent to its Euclidean counterpart. If (n_j) is bounded then clearly G is a uniform domain, and hence a Slice domain. Suppose (n_j) is unbounded. We shall show that G cannot even satisfy a uniform Slice condition for z, z_0, where $z_0 = (3/4, 0)$ and $z = ((t_k + t_{k+1})/2, 0)$ for arbitrary $k \in \mathbb{N}$. If a C-Slice condition holds then the path γ for the pair z, z_0 meets the line $x = t_k$ at some point z'. Now z' lies in some slice S_i. We must have $\operatorname{dia} S_i \gtrsim t_k/C$: this follows from part (iv) of the Slice definition if $i = 0$ or $i = j = j(k)$, and from the fact that S_i separates z and z_0 for all other i. On the other hand, $\delta_G(z') \le t_k/n_k$. Thus, part (iv) of the Slice definition implies that $n_j \lesssim C^2$, as required. The converse direction is easy.

It remains to show that G is an α-mCigar domain. We first wish to construct auxiliary subdomains G_k, $k \in \mathbb{N}$. Let $L_{k,i}$ and $z_{k,i}$, $i = 0, \dots, n_k$, be an enumeration of the (line segment) components of $G \cap (\{t_k\} \times \mathbb{R})$ and their midpoints, let $D_{k,i}$ be the intersection of the strip $\{(u,v) : t_k + t_{k+1} < 2u < t_k + t_{k-1}\}$ with the disk that has $L_{k,i}$ as a diameter, and let

$$G_k = \{(u,v) \in G \,:\, t_k < u < t_{k-1}\} \cup \left(\bigcup_{i=0}^{n_k} D_{k,i} \right).$$

It is not hard to show that G_k is a C-uniform domain for some universal constant C, and so G_k is an (α, C')-mCigar domain for some $C' \lesssim 1$.

If a pair of points $z_i = (x_i, y_i)$, $i = 1, 2$, does not lie in a single subdomain G_k, we can deduce an α-mCigar condition for them by combining the conditions for intermediate sets of the form G_k. To see this, we assume without loss of generality that $t_{j+1} < x_2 \le t_j \le t_i < x_1 \le t_{i-1}$ for some $i \le j$. For each k satisfying $i - 1 \le k \le j + 1$, we choose points w_k such that $w_{i-1} = z_1$, $w_{j+1} = z_2$, and w_k is one of the points $z_{k,i}$ for each $i \le k \le j$. If $i < k < j$, it does not matter which of these points we pick but we pick w_i and w_j so as to minimize $|w_{i-1} - w_i|$ and $|w_{j+1} - w_j|$. The desired α-mCigar condition follows from the triangle inequality and the geometric decay of the $d_{\alpha,G}$-distances from w_k to w_{k+1}. $\square$

Finally, we consider the class of *Steiner symmetric cusp domains*

$$SC(\phi,n) \equiv \{x=(x_1,x') \in \mathbb{R}\times\mathbb{R}^{n-1} \,:\, |x'|<\phi(x_1),\, 0<x_1<1\},$$

where $\phi:(0,1)\to(0,\infty)$ is a strictly increasing continuous function which satisfies $\phi(0)=0$ and $\liminf_{t\to 0+}\phi(t)/t=0$. We shall see that these cusp domains never support p-Hölder imbeddings. Note that if instead the above lim inf were positive and $|\phi(t_1)-\phi(t_2)|\gtrsim|t_2-t_1|$ for all $0<t_1,t_2<t_0$, then the boundary of the domain $SC(\phi,n)$ would locally be the graph of a Lipschitz function and thus all of the Hölder imbeddings would be valid on $SC(\phi,n)$ by the classical theorem (which, by the way, is due to Morrey [Mo]).

Proposition 4.6. *Suppose that $U=SC(\phi,n)$ is a Steiner symmetric cusp domain and that $p>n$. Then U does not support an (inner) p-Hölder imbedding, while $B(0,2)\setminus\overline{U}$ supports a p-Hölder imbedding if and only if $n>2$, and it supports an inner p-Hölder imbedding in all dimensions.*

The fact that cusp domains do not support Hölder imbeddings is not very new (for instance it is implicit in [KR, 5.2]), but our approach is different.

Proof of Proposition 4.6. Let $\alpha=(p-n)/(p-1)$. We first consider the non-imbedding result for U itself. We claim that pairs of points on the x_1-axis of U satisfy an (α,C)-wSlice condition for some $C=C(p,n)$. If $n=2$, then U is conformally equivalent to the unit disk; the claim then follows from Lemma 2.9. If $n>2$, then the associated planar domain $U_2=SC(\phi,2)$ is an α-wSlice domain. Take arbitrary points $x=(x_1,0)$ and $y=(y_1,0)\in U_2$, and let γ, $\{S_i,d_i\}_{i=1}^m$ be the associated slice data. By Lemma 2.5 we may assume that $d_i=\mathrm{dia}(S_i)$. We may also assume that $\gamma=[x\to y]$, since this is the minimal d_{α,U_2}-length path for the pair x,y. Let $X,Y\in U$ be defined by $X=(x_1,0)$ and $Y=(y_1,0)$; of course "0" is now an $(n-1)$-dimensional vector. The slice data for x,y induce slice data γ', $\{T_i,e_i\}_{i=1}^m$ for X,Y: take $\gamma'=[X\to Y]$, $T_i=\{(u_1,u')\,:\,(u_1,|u'|)\in S_i\}$, and $e_i=\mathrm{dia}(T_i)$. Since each S_i intersects the x_1-axis, it is not hard to see that $e_i\approx d_i$. With this in hand, it is a routine matter to verify the slice properties.

In light of the α-wSlice property that we have proved and the converse part of Theorem 3.2, we need only check that pairs of points on the x_1-axis of U can be found that fail to satisfy any given (α,C)-mCigar condition. Consider points $x=(a,0)$, $y=(b,0)$, where $0<a<b<1$. For any $z=(z_1,0)$, we have $\delta_U(z)\le\phi(z_1)$. If x,y satisfy an (α,C)-mCigar condition, then

$$F(b)\equiv\int_0^b \phi(t)^{\alpha-1}\,dt \le Cb^\alpha, \tag{4.7}$$

for each $b\in[0,1]$. Next, let $M>0$ be arbitrary. By the assumptions on ϕ, there exists $t_M>0$ such that $\phi(t)<t_M/M$ on the entire interval $[0,t_M]$. Putting this into (4.7) would yield $M^{1-\alpha}t_M^\alpha\le Ct_M^\alpha$, which is certainly untenable for large M.

Next we consider $G\equiv B(0,2)\setminus\overline{U}$. As the reader may verify, G is an inner uniform domain in all dimensions, and is a uniform domain if $n>2$; the positive imbedding results follow. In the planar case, G is not an α-mCigar domain, as can be verified by considering two points $(x_1,\pm2\phi(x_1))$. But G an (inner) α-wSlice domain (since it is inner uniform), and so G cannot support a p-Hölder imbedding when $n=2$. $\square$

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