Heat Polynomial Generalizations

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ABSTRACT. We generalize the heat polynomials for the heat equation to more general partial differential equations, of higher order with respect to both the time variable and the space variables. Whereas the heat equation requires only one family of polynomials, for an equation of the ℓ-th order with respect to time ℓ families of polynomials are needed, corresponding to the ℓ initial conditions specified in the Cauchy problem.

1. Introduction

The classical heat polynomials $\{p_\beta(x,t)\}$ are polynomial solutions of the heat equation,

$$\frac{\partial u(x,t)}{\partial t} = \Delta_n u(x,t) \quad ,$$

satisfying the initial condition

$$p_\beta(x,0) = x^\beta \quad . \tag{1.1}$$

These polynomials appear in early work of Appell [**1**] on the heat equation, and were later investigated in detail by Rosenbloom and Widder [**8, 9, 10, 11**]. The polynomials are particularly useful in solving the Cauchy problem

$$\frac{\partial u(x,t)}{\partial t} = \Delta_n u(x,t) \quad , \quad u(x,0) = f(x) \quad . \tag{1.2}$$

If $f(x)$ has the Taylor series expansion

$$f(x) = \sum_\beta c_\beta x^\beta \quad ,$$

then, under certain growth conditions on f, a solution of (1.2) is given by the expansion

$$u(x,t) = \sum_\beta c_\beta p_\beta(x,t) \quad .$$

For discussion of other applications of the heat polynomials, see [**11**] and [**6**].

Several authors have devised analogs of the heat polynomials for specific equations besides the heat equation; prominent examples are Bragg [**2**], Haimo and Markett [**3, 4**], and Kemnitz [**7**]. A longer list of references can be found in [**5, 6**].

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In the latter two papers, the present authors establish explicit formulas for polynomial solutions $\{p_\beta\}$ of a general evolution equation,

$$\frac{\partial u(x,t)}{\partial t} = \sum_\alpha a_\alpha(t)\,\partial_x^\alpha u(x,t) \quad , \tag{1.3}$$

satisfying the initial condition (1.1). They also investigate properties of these polynomials, obtain pointwise upper bound estimates, and use these to solve Cauchy problems with use of series expansions in terms of the polynomials.

Here we present analogs of heat polynomials for an equation involving a higher order time derivative,

$$\frac{\partial^\ell u(x,t)}{\partial t^\ell} = \sum_\alpha a_\alpha \partial_x^\alpha u(x,t) \quad , \tag{1.4}$$

where ℓ may be any positive integer. But Cauchy data for this equation involves ℓ initial conditions,

$$\frac{\partial^k u(x,0)}{\partial t^k} = f_k(x) \quad , \quad 0 \le k < \ell \quad , \tag{1.5}$$

and correspondingly we require k families of polynomial solutions $\{p_{\beta,k}(x,t)\}$, $0 \le k < \ell$. The k-th family of polynomials solves the Cauchy conditions

$$\frac{\partial^j}{\partial t^j} p_{\beta,k}(x,0) = \delta_{jk} x^\beta = \begin{cases} x^\beta \ , & if\ j = k \ , \\ 0 \ , & if\ j \ne k\ and\ 0 \le j < \ell \ . \end{cases}$$

Consequently, if each f_k in (1.5) has a Taylor expansion

$$f_k(x) = \sum_\beta c_{\beta,k} x^\beta \quad ,$$

one would expect that a solution u of the Cauchy problem (1.4) – (1.5) might be expressed as the superposition

$$u(x,t) = \sum_{k=0}^{\ell-1} \sum_\beta c_{\beta,k} p_{\beta,k}(x,t) \quad . \tag{1.6}$$

We introduce the polynomials $\{p_{\beta,k}(x,t)\}$ as the coefficients in the expansion in powers of y of generating functions $\{G_k(x,t,y)\}$, and we use this expansion to develop a few of their properties. The generating functions have explicit representations in terms of the coefficients of the differential equation. Thereby we are able to derive explicit formulas for the polynomials in terms of these coefficients.

2. The Equation

Let $\mathcal{L}$ denote the linear differential operator, with constant coefficients $\{a_\alpha\}$,

$$\mathcal{L}u = \sum_\alpha a_\alpha \partial_x^\alpha u \quad . \tag{2.1}$$

Given a positive integer ℓ, we consider the differential equation

$$\frac{\partial^\ell u}{\partial t^\ell} = \mathcal{L}u \quad . \tag{2.2}$$

The functions $u = u(x,t) = u(x_1, x_2, \cdots, x_n, t)$ in this equation are defined for $(x,t) \in \mathbb{R}^n \times \mathbb{R}$. Although in most applications u is real valued and the coefficients $\{a_\alpha\}$ are real, our analysis is equally valid for complex valued u and complex $\{a_\alpha\}$, and so we allow this more general setting. We shall refer to t as the "time variable" and x the "space variable", although this distinction is somewhat arbitrary, as these physical interpretations do not pertain to all such equations. For multi-indices $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ in $\mathbb{R}^n$ we adopt the usual notation

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n \quad , \quad \alpha! = \alpha_1!\alpha_2! \cdots \alpha_n! \quad , \quad x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \quad ,$$

and we let ∂_x^α denote the "space derivative"

$$\partial_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}} \quad .$$

We assume the summation in (2.1) is over only a finite number of multi-indices α. At times it is convenient to number these multi-indices as $\alpha^1, \alpha^2, \cdots, \alpha^I$, and the corresponding coefficients $\{a_\alpha\}$ as $a_1, a_2, \cdots, a_I$, where $I \geq 1$, so that $\mathcal{L}$ may be written in the alternate formulation

$$\mathcal{L}u = \sum_{i=1}^{I} a_i \partial_x^{\alpha^i} u \quad . \tag{2.3}$$

3. The Function E

We will define generating functions for polynomial solutions of (2.2) in terms of a more elementary function,

$$E(s, t; \ell, k) = \sum_{m=0}^{\infty} \frac{s^m t^{\ell m + k}}{(\ell m + k)!} \quad . \tag{3.1}$$

We view ℓ and k as parameters of this function, and s and t as the independent variables. We restrict ℓ to belong to the set $\mathbb{N}$ of positive integers, and k to the set $\mathbb{N}_0$ of nonnegative integers. We allow s to be any complex number, although in most applications it will be real. In order to avoid complex derivatives we restrict t to be real, although much of what we do is equally valid if t is complex. It is clear that the power series (3.1) converges for all values of $(s, t, \ell, k) \in \mathbb{C} \times \mathbb{R} \times \mathbb{N} \times \mathbb{N}_0$, representing a C^∞ function with respect to the variables s and t, and with termwise differentiation of all orders with respect to s and/or t valid. The function E might be regarded as somewhat of a generalized exponential function. Observe that

$$\begin{aligned} E(1, t; 1, 0) &= e^t \quad , \quad & E(s, t; 1, 0) &= e^{st} \quad , \\ E(1, t; 2, 0) &= \cosh t \quad , \quad & E(1, t; 2, 1) &= \sinh t \quad , \\ E(-1, t; 2, 0) &= \cos t \quad , \quad & E(-1, t; 2, 1) &= \sin t \quad . \end{aligned}$$

From (3.1) it follows that, for $k = 1, 2, 3, \cdots$,

$$\frac{\partial}{\partial t} E(s, t; \ell, k) = E(s, t; \ell, k-1) \quad , \quad \int_0^t E(s, r; \ell, k-1)\, dr = E(s, t; \ell, k) \quad .$$

We are mainly interested in integral values of ℓ and k such that $0 \leq k < \ell$. Writing (3.1) in the expanded form

$$E(s, t; \ell, k) = \left[\frac{t^k}{k!} + \frac{s t^{\ell + k}}{(\ell + k)!} + \frac{s^2 t^{2\ell + k}}{(2\ell + k)!} + \frac{s^3 t^{3\ell + k}}{(3\ell + k)!} + \cdots \right] \quad ,$$

we determine readily that, for $0 \leq k < \ell$,

$$\frac{\partial^j}{\partial t^j} E(s, 0; \ell, k) = \begin{cases} 1 \ , & \textit{if } j = k \ , \\ 0 \ , & \textit{if } j \neq k \textit{ and } 0 \leq j < \ell \ . \end{cases} \tag{3.2}$$

Moreover, also for $0 \le k < \ell$,

$$\frac{\partial^\ell}{\partial t^\ell} E\left(s,t;\ell,k\right) = \sum_{m=1}^{\infty} \frac{s^m t^{\ell m+k-\ell}}{(\ell m+k-\ell)!} = \sum_{m=0}^{\infty} \frac{s^{m+1} t^{\ell m+k}}{(\ell m+k)!} \quad ,$$

and thus

$$\frac{\partial^\ell}{\partial t^\ell} E\left(s,t;\ell,k\right) = sE\left(s,t;\ell,k\right) \quad , \quad 0 \le k < \ell \ . \tag{3.3}$$

4. The Polynomials

It is useful to designate a "vector of coefficients", associated with $\mathcal{L}$ and more specifically with its alternate form (2.3),

$$a = \left(a_1, a_2, \cdots, a_I\right) \quad , \tag{4.1}$$

as well as an associated polynomial $Q = Q\left(y\right) = Q\left(y_1, \cdots, y_n\right)$, $y \in \mathbb{R}^n$,

$$Q\left(y\right) = \sum_{\alpha} a_\alpha y^\alpha = \sum_{i=1}^{I} a_i y^{\alpha^i} \quad . \tag{4.2}$$

Given a positive integer ℓ, we employ ℓ different "generating functions" associated with the operator $\mathcal{L}$, labelled as $\{G_k\}$, $k = 0, 1, \cdots, \ell-1$, and defined for $(x,t,y) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ according to

$$\begin{aligned} G_k\left(x,t,y\right) &= e^{x\cdot y} E\left(Q\left(y\right), t; \ell, k\right) \\ &= e^{x\cdot y} \sum_{m=0}^{\infty} \frac{Q\left(y\right)^m t^{\ell m+k}}{(\ell m+k)!} \quad (0 \le k < \ell) \quad . \end{aligned} \tag{4.3}$$

(It would perhaps be more precise to subscript G as $G_{\ell,k}$, but we wish to keep the notation uncluttered and hence from now on consider ℓ as fixed.) The notation $x \cdot y$ denotes the usual dot product in $\mathbb{R}^n$; it is easily checked that

$$e^{x\cdot y} = \sum_{\gamma} \frac{x^\gamma y^\gamma}{\gamma!} \quad , \tag{4.4}$$

with the summation over all multi-indices $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ in $\mathbb{R}^n$.

We examine some properties of the generating functions $\{G_k\}$. An immediate consequence of (4.3) is the pair of formulas, valid for $1 \le k < \ell$,

$$\frac{\partial}{\partial t} G_k\left(x,t,y\right) = G_{k-1}\left(x,t,y\right) \quad , \quad \int_0^t G_{k-1}\left(x,r,y\right)\, dr = G_k\left(x,t,y\right) \quad .$$

From (3.2) we infer that, for $0 \le k < \ell$,

$$\frac{\partial^j G_k\left(x,0,y\right)}{\partial t^j} = \begin{cases} e^{x\cdot y} \ , & if\ j = k \ , \\ 0 \ , & if\ j \ne k\ and\ 0 \le j < \ell \ . \end{cases} \tag{4.5}$$

Moreover, from (3.3) and (4.3),

$$\frac{\partial^\ell}{\partial t^\ell} G_k\left(x,t,y\right) = Q\left(y\right) G_k\left(x,t,y\right) = \sum_{\alpha} a_\alpha \partial_x^\alpha G_k\left(x,t,y\right) \quad ;$$

that is, for each fixed y in $\mathbb{R}^n$,

$$\frac{\partial^\ell}{\partial t^\ell} G_k\left(x,t,y\right) = \mathcal{L} G_k\left(x,t,y\right) \quad , \quad 0 \le k < \ell \ . \tag{4.6}$$

We shall write each G_k in the expanded form

$$G_k(x,t,y) = \sum_{\beta} p_{\beta,k}(x,t)\,\frac{y^\beta}{\beta!} \quad , \tag{4.7}$$

where for each $k = 0, 1, \cdots, \ell - 1$, the collection $\{p_{\beta,k}(x,t)\}$ is a family of functions indexed by multi-indices β in $\mathbb{R}^n$.

We recall the general multinomial formula,

$$(c_1 + c_2 + \cdots + c_I)^m = \sum_{|\sigma|=m} \frac{m!}{\sigma!}\, c^\sigma \quad ,$$

where $c = (c_1, c_2, \cdots, c_I)$, $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_I)$, and the summation is over all multi-indices σ in R^I of magnitude m. With this formula, (4.2) leads to

$$\begin{aligned} Q(y)^m = \left(\sum_{i=1}^{I} a_i y^{\alpha^i}\right)^m &= \sum_{|\sigma|=m} \frac{m!}{\sigma!} \left(a_1 y^{\alpha^1}, a_2 y^{\alpha^2}, \cdots, a_I y^{\alpha^I}\right)^\sigma \\ &= \sum_{|\sigma|=m} \frac{m!}{\sigma!}\, a_1^{\sigma_1} a_2^{\sigma_2} \cdots a_n^{\sigma_n}\; y^{\sigma_1\alpha^1 + \sigma_2\alpha^2 + \cdots + \sigma_I \alpha^I} \quad . \end{aligned}$$

We define a "vector of multi-indices"

$$\overline{\alpha} = \left(\alpha^1, \alpha^2, \cdots, \alpha^I\right) \quad ,$$

and introduce a "dot product"

$$\overline{\alpha}\cdot\sigma = \sigma_1\alpha^1 + \sigma_2\alpha^2 + \cdots + \sigma_I\alpha^I \quad , \tag{4.8}$$

so that $Q(y)^m$ can be written more briefly as

$$Q(y)^m = \sum_{|\sigma|=m} \frac{m!}{\sigma!}\, a^\sigma\, y^{\overline{\alpha}\cdot\sigma} \quad .$$

(Note that $\overline{\alpha}\cdot\sigma$ is a multi-index in $\mathbb{R}^n$.) Substitution of this expression into (4.3), with use also of (4.4), gives

$$\begin{aligned} G_k(x,t,y) &= \sum_\gamma \frac{x^\gamma y^\gamma}{\gamma!} \sum_{m=0}^{\infty} \frac{t^{\ell m + k}}{(\ell m + k)!} \sum_{|\sigma|=m} \frac{m!}{\sigma!}\, a^\sigma\, y^{\overline{\alpha}\cdot\sigma} \\ &= \sum_\gamma \sum_\sigma \frac{x^\gamma y^\gamma}{\gamma!} \frac{t^{\ell|\sigma|+k}}{(\ell|\sigma|+k)!} \frac{|\sigma|!\, a^\sigma y^{\overline{\alpha}\cdot\sigma}}{\sigma!} \quad , \end{aligned} \tag{4.9}$$

with the summation over all multi-indices γ in $\mathbb{R}^n$ and σ in $\mathbb{R}^I$.

We want to fully justify any rearrangements and termwise differentiations of the double series (4.9) for G_k. We look at a compact region where $|x|, |y|, |t| \le M$, with $M \ge 1$. Applying estimates of the type $|x^\alpha| \le |x|^{|\alpha|}$, for each term of (4.9) we have the bound

$$\left| \frac{x^\gamma y^\gamma}{\gamma!} \frac{t^{\ell|\sigma|+k}}{(\ell|\sigma|+k)!} \frac{|\sigma|!\, a^\sigma y^{\overline{\alpha}\cdot\sigma}}{\sigma!} \right| \le \frac{M^{|\gamma|} M^{|\gamma|}}{\gamma!} \frac{|\sigma|!}{(\ell|\sigma|+k)!} \frac{M^{\ell|\sigma|+k}\, |a|^{|\sigma|}\, M^{|\overline{\alpha}\cdot\sigma|}}{\sigma!} \quad .$$

Let L denote the maximum order of any space derivative in (2.1), so that $|\alpha| \le L$ for each α. Then

$$|\overline{\alpha}\cdot\sigma| = \sum_{i=1}^{I} \left|\alpha^i\right| \sigma_i \le L \sum_{i=1}^{I} \sigma_i = L|\sigma| \quad , \quad M^{|\overline{\alpha}\cdot\sigma|} \le M^{L|\sigma|} \quad .$$

Using also the crude estimate

$$\frac{|\sigma|!}{(\ell\,|\sigma|+k)!}\leq\frac{1}{k!}\quad,$$

along with the general formula

$$\sum_{\beta}\frac{r^{|\beta|}}{\beta!}=e^{nr}\quad,$$

valid for real numbers r with the sum over multi-indices β in $\mathbb{R}^n$, we find that (4.9) is majorized by

$$\begin{aligned}&\sum_{\gamma}\sum_{\sigma}\left|\frac{x^{\gamma}y^{\gamma}}{\gamma!}\frac{t^{\ell|\sigma|+k}}{(\ell\,|\sigma|+k)!}\frac{|\sigma|!a^{\sigma}y^{\overline{\alpha}\cdot\sigma}}{\sigma!}\right|\\ &\leq\frac{M^k}{k!}\sum_{\gamma}\frac{M^{2|\gamma|}}{\gamma!}\sum_{\sigma}\frac{|a|^{|\sigma|}M^{\ell|\sigma|}M^{L|\sigma|}}{\sigma!}\\ &\leq\exp M\exp\left(nM^2\right)\exp\left(I\,|a|\,M^{\ell+L}\right)\quad.\end{aligned}$$

Thus (4.9) converges absolutely and uniformly in any compact region containing the variables x, t, and y.

With similar estimates we can verify that any differentiated series of (4.9), of any order and with respect to any combination of components of the variables x, t, y, likewise converges absolutely and uniformly in any compact region. (Any differentiated series will be majorized by a series comparable to an exponential series via the ratio test.) In particular, orders of summation can be freely interchanged in (4.9), with termwise differentiation also legitimate.

Now we interchange orders of summation in (4.9), first summing outside over powers y^{β}, as $\beta=\gamma+\overline{\alpha}\cdot\sigma$ ranges over all multi-indices in $\mathbb{R}^n$, getting

$$G_k\left(x,t,y\right)=\sum_{\beta}y^{\beta}\sum_{\gamma+\overline{\alpha}\cdot\sigma=\beta}\frac{|\sigma|!}{(\ell\,|\sigma|+k)!}\frac{a^{\sigma}x^{\gamma}t^{\ell|\sigma|+k}}{\gamma!\sigma!}\quad.\tag{4.10}$$

Comparing this representation with (4.7), we find that our formula for $p_{\beta,k}$ is

$$p_{\beta,k}\left(x,t\right)=\beta!\sum_{\gamma+\overline{\alpha}\cdot\sigma=\beta}\frac{|\sigma|!}{(\ell\,|\sigma|+k)!}\frac{a^{\sigma}x^{\gamma}t^{\ell|\sigma|+k}}{\gamma!\sigma!}\qquad(0\leq k<\ell)\ .\tag{4.11}$$

Given multi-indices α and β in $\mathbb{R}^n$ we say that $\alpha\leq\beta$ provided that $\alpha_i\leq\beta_i$ for each i. The sum in (4.11) is over multi-indices γ in $\mathbb{R}^n$ and σ in $\mathbb{R}^I$ satisfying the condition $\gamma+\alpha\cdot\sigma=\beta$. But the only way this condition is possible is that $\overline{\alpha}\cdot\sigma\leq\beta$ and $\gamma=\beta-\overline{\alpha}\cdot\sigma$, and for any such σ there is only one corresponding γ. Thus we may rewrite (4.11) in the equivalent formulation

$$p_{\beta,k}\left(x,t\right)=\beta!\sum_{\overline{\alpha}\cdot\sigma\leq\beta}\frac{|\sigma|!}{(\ell\,|\sigma|+k)!}\frac{a^{\sigma}x^{\beta-\overline{\alpha}\cdot\sigma}t^{\ell|\sigma|+k}}{(\beta-\overline{\alpha}\cdot\sigma)!\sigma!}\qquad(0\leq k<\ell)\ ,\tag{4.12}$$

with the summation now taken over all multi-indices σ in $\mathbb{R}^I$ such that $\overline{\alpha}\cdot\sigma\leq\beta$.

As termwise differentiation is permissible, (4.6) and (4.7) give

$$\sum_{\beta}\frac{\partial^{\ell}}{\partial t^{\ell}}p_{\beta,k}\left(x,t\right)\frac{y^{\beta}}{\beta!}=\frac{\partial^{\ell}}{\partial t^{\ell}}G_k\left(x,t,y\right)=\mathcal{L}G_k\left(x,t,y\right)=\sum_{\beta}\mathcal{L}p_{\beta,k}\left(x,t\right)\frac{y^{\beta}}{\beta!}\quad,$$

and hence

$$\frac{\partial^\ell}{\partial t^\ell} p_{\beta,k}(x,t) = \mathcal{L} p_{\beta,k}(x,t) \quad . \tag{4.13}$$

Also, from (4.7) and (4.5), if $0 \le k < \ell$ then

$$\begin{aligned}\sum_\beta \left[\frac{\partial^j}{\partial t^j} p_{\beta,k}(x,0)\right] \frac{y^\beta}{\beta!} &= \frac{\partial^j G_k(x,0,y)}{\partial t^j} \\ &= \begin{cases} e^{x\cdot y} = \sum_\beta \frac{x^\beta y^\beta}{\beta!} \ , & if\ j = k \ , \\ 0 \ , & if\ j \ne k\ and\ 0 \le j < \ell \ , \end{cases}\end{aligned}$$

from which we conclude, for $0 \le k < \ell$,

$$\frac{\partial^j}{\partial t^j} p_{\beta,k}(x,0) = \begin{cases} x^\beta \ , & if\ j = k \ , \\ 0 \ , & if\ j \ne k\ and\ 0 \le j < \ell \ . \end{cases} \tag{4.14}$$

As there are only a finite number of powers x^γ with $\gamma \le \beta$, (4.11) shows that $p_{\beta,k}(x,t)$ is a polynomial in x when t is held fixed. If the operator $\mathcal{L}$ of (2.1) contains a *zero order term* – that is, if some nonzero a_α appears corresponding to $\alpha = (0, \cdots, 0)$ – then the summations in (4.11) and (4.12) will have an infinite number of terms, as there will be an infinite number of multi-indices σ in $\mathbb{R}^I$ with $\overline{\alpha} \cdot \sigma \le \beta$. (A term $\sigma_i \alpha^i$ in (4.8) contributes nothing to the sum if $\alpha^i = (0, \cdots, 0)$.) Thus, if $\mathcal{L}$ has a zero order term, $p_{\beta,k}(x,t)$ need not be a polynomial with respect to t. However, if $\mathcal{L}$ has no zero order term then the number of multi-indices σ with $\overline{\alpha} \cdot \sigma \le \beta$ is finite, and (4.11) and (4.12) both are finite summations; in this event $p_{\beta,k}(x,t)$ is a polynomial in both x and t; that is, it is a polynomial in the $n+1$ variables $(x_1, \cdots, x_n, t)$.

We summarize formally the results thus far of this section :

Theorem 1. *Let $\mathcal{L}$ be the operator (2.1) with real or complex constant coefficients $\{a_\alpha\}$, and let ℓ be a positive integer. Then the functions $\{p_{\beta,k}(x,t)\}$, as defined by (4.11) or the equivalent (4.12), and indexed by multi-indices β in $\mathbb{R}^n$ and integers k, $0 \le k < \ell$, solve the initial value problems*

$$\frac{\partial^\ell}{\partial t^\ell} p_{\beta,k}(x,t) = \mathcal{L} p_{\beta,k}(x,t) \quad ,$$

$$\frac{\partial^j}{\partial t^j} p_{\beta,k}(x,0) = \begin{cases} x^\beta \ , & if\ j = k \ , \\ 0 \ , & if\ j \ne k\ and\ 0 \le j < \ell \ . \end{cases}$$

Each function $p_{\beta,k}$ is a polynomial in x when t is held fixed, and is a polynomial in both x and t in the event that $\mathcal{L}$ has no zero order term.

From (4.11) we have, for $1 \le k < \ell$,

$$\frac{\partial}{\partial t} p_{\beta,k}(x,t) = p_{\beta,k-1}(x,t) \quad , \quad \int_0^t p_{\beta,k-1}(x,r)\ dr = p_{\beta,k}(x,t) \quad . \tag{4.15}$$

Given any multi-index γ in $\mathbb{R}^n$, we may differentiate (4.7) and (4.3) to obtain

$$\begin{aligned}\sum_\beta \partial_x^\gamma p_{\beta,k}(x,t)\frac{y^\beta}{\beta!} &= \partial_x^\gamma G_k(x,t,y) = y^\gamma G_k(x,t,y)\\ &= y^\gamma \sum_\beta p_{\beta,k}(x,t)\frac{y^\beta}{\beta!} = \sum_\beta p_{\beta,k}(x,t)\frac{y^{\beta+\gamma}}{\beta!}\\ &= \sum_{\beta\ge\gamma} p_{\beta-\gamma,k}(x,t)\frac{y^\beta}{(\beta-\gamma)!} \quad ;\end{aligned}$$

consequently,

$$\partial_x^\gamma p_{\beta,k}(x,t) = \begin{cases} \frac{\beta!}{(\beta-\gamma)!}\, p_{\beta-\gamma,k}(x,t) \quad , & if\ \beta\ge\gamma \ . \\ 0 \quad , & if\ \beta\not\ge\gamma \ . \end{cases} \tag{4.16}$$

5. Examples

We give a few examples of equations having the form (2.2), and of the associated polynomials.

EXAMPLE 1. *If we take $\ell = 1$ and $\mathcal{L} = \Delta_n$, then (2.2) becomes the heat equation in n space variables,*

$$\frac{\partial u}{\partial t} = \Delta_n u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} \quad .$$

We have $a = (1,\cdots,1)$, $\overline{\alpha} = (2e_1,\cdots,2e_n)$, where e_i denotes the i-th unit coordinate vector; also, $\sigma = (\sigma_1,\cdots,\sigma_n)$ and $\overline{\alpha}\cdot\sigma = 2\sigma_1 e_1+\cdots+2\sigma_n e_n = 2\sigma$. In formula (4.12) we must take $k = 0$, and thus we obtain but one family of polynomials,

$$p_\beta(x,t) = \beta! \sum_{2\sigma\le\beta} \frac{x^{\beta-2\sigma}t^{|\sigma|}}{(\beta-2\sigma)!\sigma!} \quad .$$

These are the classical n-dimensional heat polynomials studied by Rosenbloom and Widder [**8, 9, 10**].

EXAMPLE 2. *If we take $\ell = 1$ but leave L as in (2.1), then (2.2) simplifies to an equation studied by the present authors in* [**5, 6**]. *Again, in (4.12) we must take $k = 0$, obtaining the single family of polynomials*

$$p_\beta(x,t) = \beta! \sum_{\overline{\alpha}\cdot\sigma\le\beta} \frac{a^\sigma x^{\beta-\overline{\alpha}\cdot\sigma}t^{\ell|\sigma|+k}}{(\beta-\overline{\alpha}\cdot\sigma)!\sigma!} \quad .$$

Each polynomial p_β solves the Cauchy problem

$$\frac{\partial}{\partial t}p_\beta(x,t) = \mathcal{L}p_\beta(x,t) \quad , \quad p_\beta(x,0) = x^\beta \ .$$

EXAMPLE 3. *With $\ell = 1$ and in one space dimension $n = 1$, and with $\mathcal{L} = \partial^r/\partial x^r$, $r\ge 1$, (2.2) becomes the equation studied by Kemnitz* [**7**],

$$\frac{\partial u}{\partial t} = \frac{\partial^r u}{\partial x^r} \quad .$$

Formula (4.11) specializes to the formula of Kemnitz,

$$p_\beta(x,t) = \beta! \sum_{\gamma+r\sigma=\beta} \frac{x^\gamma t^\sigma}{\gamma!\sigma!} \quad ,$$

where in one space dimension the multi-indices β, γ, and σ reduce to nonnegative integers. Haimo and Markett [**3, 4**] *studied polynomial solutions of a closely related equation,*

$$\frac{\partial u}{\partial t} = (-1)^{m+1} \frac{\partial^{2m} u}{\partial x^{2m}} \quad .$$

For this equation, (4.12) specializes to the formula of Haimo and Markett,

$$p_\beta(x,t) = \beta! \sum_{2m\sigma \leq \beta} (-1)^{(m+1)\sigma} \frac{x^{\beta - 2m\sigma} t^\sigma}{\sigma! \, (\beta - 2m\sigma)!} \quad ,$$

where once again β and σ are nonnegative integers.

Example 4. *For the wave equation in n space dimensions,*

$$\frac{\partial^2 u}{\partial t^2} = \Delta_n u \quad ,$$

we have $\ell = 2$, and (4.11) gives two families of polynomials,

$$p_{\beta,0}(x,t) = \beta! \sum_{\gamma + 2\sigma = \beta} \frac{|\sigma|!}{(2|\sigma|)!} \frac{x^\gamma t^{2|\sigma|}}{\gamma! \sigma!} \quad ,$$

$$p_{\beta,1}(x,t) = \beta! \sum_{\gamma + 2\sigma = \beta} \frac{|\sigma|!}{(2|\sigma| + 1)!} \frac{x^\gamma t^{2|\sigma| + 1}}{\gamma! \sigma!} \quad ,$$

where the sum is over multi-indices γ, σ in $\mathbb{R}^n$. These polynomials solve the wave equation, with initial conditions

$$\begin{array}{ll} p_{\beta,0}(x,0) = x^\beta \quad , & p_{\beta,1}(x,0) = 0 \quad , \\ \frac{\partial}{\partial t} p_{\beta,0}(x,0) = 0 \quad , & \frac{\partial}{\partial t} p_{\beta,1}(x,0) = x^\beta \quad . \end{array}$$

In the case of one space dimension $n = 1$, both γ and σ are nonnegative integers, and it can be verified that the polynomials simplify to

$$p_{\beta,0}(x,t) = \frac{1}{2} \left[(x+t)^n + (x-t)^n \right] \quad ,$$

$$p_{\beta,1}(x,t) = \frac{1}{2(n+1)} \left[(x+t)^{n+1} - (x-t)^{n+1} \right] \quad .$$

Example 5. *The complex Cauchy-Riemann equation for analytic functions is*

$$f_x + i f_y = 0 \quad ,$$

satisfied by analytic functions $f = f(x,y)$. We write this equation in the form

$$f_y = i f_x \quad ,$$

and substitute into (4.11) with t replaced by y. We have $\ell = 1$, $k = 0$, and all multi-indices are scalars, with $\overline{\alpha} \cdot \sigma = \sigma$, $a^\sigma = i^\sigma$. Writing $p_\beta(x,y) = p_{\beta,0}(x,y)$, we find that

$$p_\beta(x,y) = \beta! \sum_{\gamma + \sigma = \beta} \frac{i^\sigma x^\gamma y^\sigma}{\gamma! \sigma!} = \sum_{\gamma=0}^{\beta} \binom{\beta}{\gamma} x^\gamma (iy)^{\beta - \gamma} = (x + iy)^\beta \quad .$$

These analytic polynomials satisfy the initial conditions $p_\beta(x,0) = x^\beta$.

Example 6. *As an example of an equation with a zero order term, consider in one space dimension a special case of the telegraph equation,*

$$\frac{\partial^2 u}{\partial t^2} = Bu + \frac{\partial^2 u}{\partial x^2} \quad .$$

The two multi-indices α *corresponding to space derivatives reduce to scalars,* $\alpha^1 = 0$ *and* $\alpha^2 = 2$*. In (4.12) also* β *and* γ *are scalars, while* $\sigma = (i, j)$ *is a multi-index in* $\mathbb{R}^2$*, and* $a = (B, 1)$ *a vector in* $\mathbb{R}^2$*, with* $\overline{\alpha} \cdot \sigma = 2j$ *and* $a^\sigma = B^i$*. Setting* $\ell = 2$*, for* $k = 0$ *and* $k = 1$ *we may write (4.12) as*

$$p_{\beta,k}(x,t) = \beta! \sum_{i,j:2j \le \beta} \frac{(i+j)!}{(2i+2j+k)!} \frac{B^i x^{\beta-2j} t^{2i+2j+k}}{(\beta-2j)! i! j!} \quad .$$

The summation is over all multi-indices (i, j) *such that* $2j \le \beta$*. As there is no restriction on* i*, we may split the sum as*

$$p_{\beta,k}(x,t) = \beta! \sum_{2j \le \beta} \frac{x^{\beta-2j}}{(\beta-2j)! j!} \sum_{i=0}^{\infty} \frac{(i+j)!}{(2i+2j+k)!} \frac{B^i t^{2i+2j+k}}{i!} \quad , \quad k = 0, 1 \quad .$$

Note that $p_{\beta,k}(x,t)$ *is a polynomial with respect to* x*, but not with respect to* t*.*

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