POLYNOMIAL SOLUTIONS TO CAUCHY PROBLEMS FOR COMPLEX BESSEL OPERATORS

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Dedicated to our friend Heinrich Begehr, in recognition of his 65th birthday.

ABSTRACT. We investigate Cauchy problems for complex differential equations of the form $\mathcal{B}u = \mathcal{L}u$, where $\mathcal{B}$ is a Bessel differential operator in the "time variable", and $\mathcal{L}$ a linear differential operator in the "space variables", possibly also involving Bessel operators. We establish conditions for existence and uniqueness of polynomial solutions whenever the Cauchy data is polynomial, and we give explicit formulas for these solutions. When the Cauchy data consists of monomials, these polynomial solutions are analogous to the heat polynomials for the heat equation.

1. Introduction

A *Bessel operator* has the form

$$\mathcal{B} = \sum_{i=0}^{\ell} \frac{b_i}{t^{\ell-i}} \frac{\partial^i}{\partial t^i} \quad ,$$

where the coefficients $\{b_i\}$ are complex constants. The "time variable" t may be either a real or complex variable, with the derivative $\partial/\partial t$ correspondingly either an ordinary derivative or a complex derivative. We seek polynomial solutions of Cauchy problems of the form

$$\mathcal{B}u\,(x,t) = \mathcal{L}u\,(x,t) \quad , \tag{1.1}$$

$$\frac{\partial^i u\,(x,0)}{\partial t^i} = \delta_{ik}\, q\,(x) \qquad , \qquad 0 \le i < \ell \quad , \tag{1.2}$$

where $k \in \mathbb{Z}$, $0 \le k < \ell$, and q is a polynomial in $x = (x_1, \cdots, x_n)$. The "space variables" $\{x_i\}$ likewise may be real or complex, and the linear operator $\mathcal{L}$ may involve Bessel operators with respect to these variables. Of particular interest is the case $q\,(x) = x^\beta$, when polynomial solutions of the problem can be regarded as analogues of the classical *heat polynomials*. (The designations "time variable" and "space variables" are somewhat arbitrary, as there is no requirement that the variables have these physical interpretations. However, in many applications such is the case.)

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The heat polynomials $\{p_\beta\}$ are polynomial solutions of Cauchy problems for the heat equation,

$$\frac{\partial}{\partial t}\, p_\beta(x,t) = \Delta p_\beta(x,t) \qquad , \qquad p_\beta(x,0) = x^\beta \quad , \tag{1.3}$$

where Δ is the Laplace operator in n space dimensions, β is a multi-index in $\mathbb{R}^n$, and

$$x^\beta = {x_1}^{\beta_1}{x_2}^{\beta_2}\ldots{x_n}^{\beta_n} \quad .$$

These polynomials appear in early work of Appell [1], and were later studied in detail by Rosenbloom and Widder [26, 28, 29, 30]. They are of interest because there are explicit formulas for them, and they are helpful in analyzing the more general Cauchy problem

$$\frac{\partial}{\partial t}\, u(x,t) = \Delta u(x,t) \qquad , \qquad u(x,0) = f(x) \quad . \tag{1.4}$$

Indeed, if f can be suitably approximated by polynomials, then under certain conditions a solution u of (1.4) can be approximated by linear combinations of heat polynomials. Various authors have produced analogues of the heat polynomials for more general partial differential equations. We cite several examples in this paper, and refer the reader to [15, 16, 17, 18] for further references.

In Section 2 of this paper we discuss the factorization of Bessel operators. In Section 3 we determine conditions for existence and uniqueness of polynomial solutions of problem (1.1) – (1.2), and produce explicit formulas for these solutions; these results are quite general in nature. In Section 4 we specialize to $q(x) = x^\beta$ and to space operators of the form

$$\mathcal{L}u = \sum_\alpha a_\alpha \partial^\alpha \quad .$$

In this case a *generating function* can be displayed for the polynomial solutions, which we describe in Sections 5 and 6. In Section 7 we specify that $\mathcal{L}$ be a Bessel operator in one space variable. Then certain restrictions must be placed on the Cauchy data $q(x)$ and x^β in order to guarantee polynomial solutions. The theory in this case can be specialized to produce results of several authors, as we point out. Finally, in Section 8 we extend the analysis of Section 7 to the most general version of (1.1), allowing $\mathcal{L}$ to involve Bessel operators in all n space variables $x_1, \ldots, x_n$.

Good sources of examples and references regarding polynomial solutions of equations involving Bessel operators are the papers of Bragg and Dettman [2, 3, 5, 6].

There is an interesting body of work on the problem of determining all polynomial solutions of systems of partial differential equations with constant coefficients, as well as the dimensions of solution spaces of polynomials of specified degree. For important papers and further bibliographical references, see [19, 20, 21, 23, 24, 25, 27].

2. Factoring Bessel Operators

We let $\mathcal{B}$ denote the ℓ-th order *Bessel operator* ($\ell \geq 1$),

$$\mathcal{B} = \frac{\partial^\ell}{\partial t^\ell} + \frac{b_{\ell-1}}{t}\frac{\partial^{\ell-1}}{\partial t^{\ell-1}} + \frac{b_{\ell-2}}{t^2}\frac{\partial^{\ell-2}}{\partial t^{\ell-2}} + \cdots + \frac{b_1}{t^{\ell-1}}\frac{\partial}{\partial t} + \frac{b_0}{t^\ell} \quad . \tag{2.1}$$

The coefficients $\{b_i : i = 0, 1, \cdots, \ell\}$ are complex constants with $b_\ell = 1$. The *auxiliary polynomial* B associated with $\mathcal{B}$, defined on complex numbers $z \in \mathbb{C}$, is

$$B(z) = b_0 + \sum_{i=1}^{\ell} b_i z (z-1)(z-2)\cdots(z-i+1) = \prod_{i=1}^{\ell}(z + \ell\nu_i) \quad , \tag{2.2}$$

where the complex numbers $\{-\ell\nu_i\}$ are the roots of B, repeated according to multiplicity. We point out that the collection $\{b_i\}$ is an *ordered* collection, as changing the numbering of these quantities will likely change the operator $\mathcal{B}$ and polynomial B. On the other hand, the collection $\{\nu_i : i = 1, 2, \ldots, \ell\}$ is *unordered*, as changing the numbering of these quantities does not affect the validity of (2.2). We will find that indeed all algebraic expressions in this paper will be symmetric with respect to the ν_i's; that is, changing the numbering of the ν_i's will not affect the value of any of these expressions. Consequently, in any such expression the choice of a numbering for the ν_i's is irrelevant.

Observe that (2.2) implies also the formula

$$b_0 + \sum_{i=1}^{\ell} b_i \left(t\frac{\partial}{\partial t}\right)\left(t\frac{\partial}{\partial t} - 1\right)\cdots\left(t\frac{\partial}{\partial t} - i + 1\right) = \prod_{i=1}^{\ell}\left(t\frac{\partial}{\partial t} + \ell\nu_i\right),$$

while an easy induction proof shows that

$$\left(t\frac{\partial}{\partial t}\right)\left(t\frac{\partial}{\partial t} - 1\right)\left(t\frac{\partial}{\partial t} - 2\right)\cdots\left(t\frac{\partial}{\partial t} - i + 1\right) = t^i\frac{\partial^i}{\partial t^i} \quad ;$$

consequently we may write $\mathcal{B}$ in the factored form

$$\mathcal{B} = \frac{1}{t^\ell}\prod_{i=1}^{\ell}\left(t\frac{\partial}{\partial t} + \ell\nu_i\right) \quad . \tag{2.3}$$

In particular, for any integer j,

$$\mathcal{B}\left(t^j\right) = t^{j-\ell}\prod_{i=1}^{\ell}(j + \ell\nu_i) = B(j)\, t^{j-\ell} \quad . \tag{2.4}$$

It is clear that, given an ordered collection $\{b_i\}$, equation (2.2) determines uniquely the unordered collection $\{\nu_i\}$. We now demonstrate how (2.2) may be used also to determine uniquely $\{b_i\}$ from $\{\nu_i\}$. Since each side of (2.2) is a polynomial of degree ℓ, this equation will be valid provided the two sides agree at $\ell+1$ distinct values of z. We temporarily denote

$$p(z) = \prod_{i=1}^{\ell} (z + \ell\nu_i) \quad ,$$

and take successively $z = 0, 1, 2, \cdots, \ell$ in (2.2) to obtain the $\ell+1$ equations

$$\sum_{k=0}^{i} \frac{i!}{(i-k)!} b_k = p(i) \quad , \quad 0 \le i \le \ell \quad . \tag{2.5}$$

A straightforward induction argument shows that (2.5) is equivalent to

$$b_i = \frac{1}{i!} \sum_{k=0}^{i} (-1)^{i-k} \binom{i}{k} p(k) \quad , \quad 0 \le i \le \ell \quad . \tag{2.6}$$

(Alternatively, it can be checked that the coefficient matrices of the linear systems (2.5) and (2.6) are inverses of one another.) Therefore,

$$b_i = \frac{1}{i!} \sum_{k=0}^{i} (-1)^{i-k} \binom{i}{k} \prod_{j=1}^{\ell} (k + \ell\nu_j) \quad , \quad 0 \le i \le \ell \quad . \tag{2.7}$$

Hence, (2.2) describes a one-to-one correspondence between ordered collections $\{b_i : 0 \le i \le \ell\}$, with $b_\ell = 1$, and unordered collections $\{\nu_i : 1 \le i \le \ell\}$. Formula (2.7) allows the recovery of the b_i's from the ν_i's.

Given a vector $w = (w_1, w_2, \cdots, w_\ell)$ and scalar c, we define the sum

$$w + c = (w_1 + c, w_2 + c, \cdots, w_\ell + c) \quad . \tag{2.8}$$

We set $\nu = (\nu_1, \nu_2, \cdots, \nu_\ell)$. We let $\{S_i : i = 0, 1, 2, \cdots, \ell\}$ denote the standard homogeneous symmetric polynomials in ℓ variables, defined for $z = (z_1, \cdots, z_\ell)$ according to

$$S_0(z) = 1 \quad , \quad S_1(z) = z_1 + z_2 + \cdots + z_\ell \quad ,$$

and for $i \ge 2$,

$$S_i(z) = \sum_{k_1 < k_2 < \cdots < k_i} z_{k_1} z_{k_2} \cdots z_{k_i} \quad .$$

Then, for any complex constant c and $z \in \mathbb{C}^\ell$,

$$\prod_{j=1}^{\ell} (c + z_j) = \sum_{j=0}^{\ell} c^{\ell-j} S_j(z) \quad . \tag{2.9}$$

In (2.7) we write $k + \ell\nu_j = (k+1) + (\ell\nu_j - 1)$ and use (2.9) to obtain

$$\begin{aligned} b_i &= \frac{1}{i!} \sum_{k=0}^{i} (-1)^{i-k} \binom{i}{k} \sum_{j=0}^{\ell} (k+1)^{\ell-j} S_j(\ell\nu - 1) \\ &= \frac{1}{i!} \sum_{j=0}^{\ell} S_j(\ell\nu - 1) \sum_{k=0}^{i} (-1)^{i-k} \binom{i}{k} (k+1)^{\ell-j} \quad . \end{aligned}$$

It is known (see [7], Formula 0.154 - 6) that

$$\sum_{k=0}^{i} (-1)^{i-k} \binom{i}{k} (k+1)^m = 0 \qquad , \qquad 0 \le m < i \quad , \quad m, i \in \mathbb{N} \quad .$$

Thus the sum over j for b_i may stop at $\ell - i$, yielding

$$b_i = \frac{1}{i!} \sum_{j=0}^{\ell-i} S_j\left(\ell\nu - 1\right) \sum_{k=0}^{i} (-1)^{i-k} \binom{i}{k} (k+1)^{\ell-j} \quad . \tag{2.10}$$

This representation demonstrates that b_i is a polynomial of degree at most $\ell - i$ with respect to $\nu = (\nu_1, \nu_2, \cdots, \nu_\ell)$, and is symmetric with respect to these variables.

3. Polynomial Solutions of Cauchy Problems

We consider the partial differential equation

$$\mathcal{B}u\left(x,t\right) = \mathcal{L}u\left(x,t\right) \quad , \tag{3.1}$$

where $\mathcal{B}$ is the Bessel operator (2.1), and $\mathcal{L}$ an operator

$$\mathcal{L} = \sum_{\alpha} a_\alpha \partial_x^\alpha \quad . \tag{3.2}$$

The complex valued function u depends on $(x,t) = (x_1, \ldots, x_n, t)$ $(n \ge 1)$. The coefficients $\{a_\alpha\}$ are complex constants, and the derivative ∂_x^α is

$$\partial_x^\alpha = \frac{\partial^{|\alpha|}}{\partial {x_1}^{\alpha_1} \partial {x_2}^{\alpha_2} \ldots \partial {x_n}^{\alpha_n}} \quad . \tag{3.3}$$

The summation (3.2) is taken over a *finite* collection of multi-indices α in $\mathbb{R}^n$, and we assume that no α is the zero multi-index; that is, the sum is a finite one and $\mathcal{L}$ has no zero order term. We allow either $t \in \mathbb{R}$ or $t \in \mathbb{C}$, and accordingly regard $\partial/\partial t$ either as an ordinary real derivative or a complex derivative. Likewise, we allow either $x \in \mathbb{R}^n$ or $x \in \mathbb{C}^n$, with (3.3) either ordinary differentiation in $\mathbb{R}^n$ or complex differentiation in $\mathbb{C}^n$. Given a complex valued polynomial $q = q(x)$, and integer k, $0 \le k < \ell$, we seek a *polynomial* solution u of (3.1), satisfying as well the Cauchy condition

$$\frac{\partial^i u\left(x,0\right)}{\partial t^i} = \begin{cases} 0 \; , & 0 \le i < \ell \; , \; i \ne k \; , \\ q\left(x\right) \; , & i = k \; , \end{cases} \tag{3.4}$$

Any polynomial u in x and t can be written in the form

$$u\left(x,t\right) = \sum_{j=0}^{\infty} p_j\left(x\right) t^j \quad ,$$

where each p_j is a polynomial in x, and only a finite number of $\{p_j\}$ are nonzero. The initial condition (3.4) requires further that

$$p_k\left(x\right) = \frac{q\left(x\right)}{k!} \qquad , \qquad p_j\left(x\right) = 0 \quad , \; 0 \le j \le \ell - 1, \; j \ne k \quad . \tag{3.5}$$

so that

$$u(x,t) = q(x)\frac{t^k}{k!} + \sum_{j=\ell}^{\infty} p_j(x)\, t^j \quad . \tag{3.6}$$

We substitute (3.6) into (3.1), while using (2.4), and obtain

$$q(x)\, B(k)\frac{t^{k-\ell}}{k!} + \sum_{j=\ell}^{\infty} p_j(x)\, B(j)\, t^{j-\ell} = \mathcal{L}q(x)\frac{t^k}{k!} + \sum_{j=\ell}^{\infty} \mathcal{L}p_j(x)\, t^j \quad . \tag{3.7}$$

Equating coefficients of like powers of t in (3.7) leads to the requirements

$$q(x)\, B(k) = 0 \quad , \tag{3.8}$$

$$p_{\ell+k}(x)\, B(\ell+k) = \frac{1}{k!}\mathcal{L}q(x) = \mathcal{L}p_k(x) \quad , \tag{3.9}$$

$$p_j(x)\, B(j) = 0 \quad , \quad \ell \le j \le 2\ell - 1 \quad , \quad j \ne k+\ell \quad , \tag{3.10}$$

$$p_j(x)\, B(j) = \mathcal{L}p_{j-\ell}(x) \quad , \quad j \ge 2\ell \quad . \tag{3.11}$$

To comply with (3.8) we stipulate that $B(k) = 0$ (necessarily, except in the single case $q \equiv 0$). If we decree further that

$$B(j) \ne 0 \quad , \quad j = \ell,\ \ell+1,\ \ell+2,\ \ldots \quad , \tag{3.12}$$

then equations (3.9) – (3.11) have *unique* solutions $\{p_j : j \ge \ell\}$, described by the recursive formulas

$$p_j(x) = \begin{cases} \mathcal{L}p_{j-\ell}(x)/B(j) \ , & j = k+\ell, k+2\ell, k+3\ell, \ldots \ , \\ 0 \ , & \text{otherwise} \ , \end{cases} \tag{3.13}$$

where p_k is prescribed by (3.5). On the other hand, if we want only the *existence* of solutions, it is sufficient to require only that

$$B(k+j\ell) \ne 0 \quad , \quad j = 1, 2, 3, \ldots \quad ; \tag{3.14}$$

then (3.13) still defines a solution of (3.9) – (3.11), although there may be other solutions as well. As the quantities $\{-\ell\nu_i\}$ are the roots of B, (3.14) may be stated alternatively as

$$k + \ell j + \ell\nu_i \ne 0 \quad , \quad i = 1, 2, \ldots, \ell \quad , \quad j = 1, 2, 3, \ldots \quad .$$

An equivalent statement is that none of the quantities

$$\{\nu_i + k/\ell : i = 1, 2, \ldots, \ell\}$$

are negative integers. Beginning with the left formula of (3.5), under assumption (3.14) we may iterate the top equation of (3.13) to deduce that

$$p_{m\ell+k}(x) = \frac{1}{k!}\frac{\mathcal{L}^m q(x)}{\prod_{j=1}^{m} B(k+j\ell)} \quad , \quad m = 1, 2, 3, \ldots \quad . \tag{3.15}$$

Then our function u of (3.6) becomes

$$u(x,t) = q(x)\frac{t^k}{k!} + \frac{1}{k!}\sum_{m=1}^{\infty}\frac{\mathcal{L}^m q(x)\, t^{m\ell+k}}{\prod_{j=1}^{m} B(k+j\ell)} \quad . \tag{3.16}$$

Since the operator $\mathcal{L}$ has no zero order term, and q is a polynomial, we have $\mathcal{L}^m q(x) = 0$ for m sufficiently large. Hence the sum in (3.16) terminates after a finite number of terms, and u is a polynomial in x and t.

We introduce some notation to manipulate the solution (3.16). For complex numbers z we employ the usual notation

$$z! = \Gamma(z+1) \quad ,$$

where Γ is the gamma function. Then $z!$ is meromorphic with poles only at the negative integers, and satisfies the functional equation

$$z! = z(z-1)(z-2)\dots(z-i+1)(z-i)! \tag{3.17}$$

whenever $z-i$ is not a negative integer. For a complex ℓ-tuple $\nu = (\nu_1, \nu_2, \dots, \nu_\ell)$ and for $c \in \mathbb{C}$ we denote

$$\nu! = \nu_1!\,\nu_2!\,\dots\,\nu_\ell! \quad , \quad \nu + c := (\nu_1 + c, \nu_2 + c, \dots, \nu_\ell + c) \quad .$$

Then, in view of (2.2), and since no value $\nu_i + k/\ell$ is a negative integer, we may write (3.15) as

$$\begin{aligned} p_{m\ell+k}(x) &= \frac{1}{k!}\frac{\mathcal{L}^m q(x)}{\prod_{j=1}^{m}\prod_{i=1}^{\ell}(k+\ell j+\ell\nu_i)} \\ &= \frac{1}{k!}\frac{\mathcal{L}^m q(x)}{\ell^{m\ell}\prod_{i=1}^{\ell}\prod_{j=1}^{m}(\nu_i+j+k/\ell)} \\ &= \frac{1}{k!}\frac{\mathcal{L}^m q(x)}{\ell^{m\ell}\prod_{i=1}^{\ell}(\nu_i+m+k/\ell)!/(\nu_i+k/\ell)!} \\ &= \frac{1}{k!}\frac{(\nu+k/\ell)!\mathcal{L}^m q(x)}{\ell^{m\ell}(\nu+m+k/\ell)!} \quad . \end{aligned}$$

By (3.5), this formula holds also for $m = 0$; thus we may write (3.16) also as

$$u(x,t) = \frac{(\nu+k/\ell)!}{k!}\sum_{m=0}^{\infty}\frac{\mathcal{L}^m q(x)\, t^{m\ell+k}}{\ell^{m\ell}(\nu+m+k/\ell)!} \quad . \tag{3.18}$$

We summarize the discussion of this section with a formal statement :

Theorem 1. *Let $\mathcal{B}$ be the Bessel operator (2.1), with auxiliary polynomial (2.2), and let $\mathcal{L}$ be the operator (3.2) with no zero order term. Let k be an integer, $0 \le k < \ell$, and suppose that*

$$B(k) = 0 \quad , \quad B(k+j\ell) \neq 0 \quad , \quad j = 1, 2, 3, \dots \quad . \tag{3.19}$$

Let $q = q(x)$ be a polynomial. Then the Cauchy problem

$$\mathcal{B}u(x,t) = \mathcal{L}u(x,t) \quad , \tag{3.20}$$

$$\frac{\partial^i u(x,0)}{\partial t^i} = \begin{cases} 0 & , 0 \le i < \ell \quad , \quad i \ne k \\ q(x) & , i = k \quad , \end{cases} \tag{3.21}$$

has a polynomial solution prescribed explicitly by (3.16) and (3.18). If moreover

$$B(j) \ne 0 \quad , \quad j = \ell,\ \ell+1,\ \ell+2,\ \ldots \quad , \tag{3.22}$$

then there is only one polynomial solution of the problem.

Remark 1. *It should be pointed out that the particular structure (3.2) of the operator $\mathcal{L}$ has not entered into the discussion. To obtain Theorem 1 and representations (3.16), (3.18), we used only linearity of $\mathcal{L}$, along with the fact that each $\mathcal{L}^m q$ is a polynomial, vanishing identically for m sufficiently large.*

Remark 2. *If in (3.16) we designate functions*

$$u_0(x,t) = \frac{q(x)\, t^k}{k!} \quad , \quad u_m(x,t) = \frac{q(x)\, t^{\ell m + k}}{k! \prod_{j=1}^m B(k + j\ell)} \qquad (m \ge 1) \quad ,$$

then we can write this formula as

$$u(x,t) = \sum_{m=0}^{\infty} \mathcal{L}^m \left[u_m(x,t) \right] \quad . \tag{3.23}$$

Moreover, with use of (3.19) and (2.4) we can verify that the sequence of functions $\{u_m\}$ has the properties

$$\mathcal{B} u_m(x,t) = \begin{cases} 0 & , \text{ if } m = 0, \\ u_{m-1}(x,t) & , \text{ if } m \ge 1. \end{cases}$$

In the terminology of Karachik [19, 21]*, the sequence $\{u_m\}$ is "0-normalized with respect to the operator $\mathcal{B}$". Karachik showed that, for a large class of constant coefficient partial differential equations of the form*

$$\mathcal{B}u - \mathcal{L}u = 0 \quad ,$$

polynomial solutions can be written in the form (3.23), where the sequence $\{u_m\}$ is 0-normalized with repect to the operator $\mathcal{B}$. Representation (3.16) is but one example of this general formulation.

Remark 3. *It appears an open question as to what operators $\mathcal{B}$, if any, besides Bessel operators one can expect polynomial solutions to Cauchy problems (3.20) – (3.21) with polynomial q.*

Obviously, for general Bessel operators $\mathcal{B}$ the conditions (3.19) need not hold, in which case Theorem 1 will not apply. For any particular operator the conditions might hold for one or more values of k, but not for others. We point out two special cases of interest.

Example 1. *Suppose the operator $\mathcal{B}$ has no zero order term, so that $b_0 = 0$ in (2.2). Then $B(0) = 0$, and (3.19) will be satisfied with $k = 0$ provided only that no root of B is a positive integral multiple of ℓ. In this event the Cauchy problem*

$$\mathcal{B}u(x,t) = \mathcal{L}u(x,t) \quad ,$$

$$\frac{\partial^i u(x,0)}{\partial t^i} = \begin{cases} q(x) & , i = 0 \quad , \\ 0 & , 1 \le i < \ell \quad , \end{cases}$$

with q a polynomial, has a polynomial solution

$$u(x,t) = q(x) + \sum_{m=1}^{\infty} \frac{\mathcal{L}^m q(x)\, t^{m\ell}}{\prod_{j=1}^{m} B(j\ell)} = \nu! \sum_{m=0}^{\infty} \frac{\mathcal{L}^m q(x)\, t^{m\ell}}{\ell^{m\ell}(\nu+m)!} \quad .$$

Example 2. *Consider the very special Bessel operator*

$$\mathcal{B} = \frac{\partial^\ell}{\partial t^\ell} \quad ,$$

with corresponding auxiliary polynomial

$$B(z) = z(z-1)(z-2)\cdots(z-\ell+1) \quad .$$

Then $B(0) = B(1) = \cdots = B(\ell-1) = 0$, and Theorem 1 applies for $k = 0, 1, \ldots, \ell-1$. The theorem ensures that for each such k the problem

$$\frac{\partial^\ell u(x,t)}{\partial t^\ell} = \mathcal{L}u(x,t) \quad ,$$

with initial condition (3.21), has only one polynomial solution. Into the representation (3.16) we substitute

$$\prod_{j=1}^{m} B(k+j\ell) = \frac{(k+m\ell)!}{k!}$$

to obtain the simplified formula

$$u(x,t) = \sum_{m=0}^{\infty} \frac{\mathcal{L}^m q(x)\, t^{m\ell+k}}{(k+m\ell)!} \quad .$$

4. Analogues of Heat Polynomials

We continue to assume the hypotheses of Theorem 1 (but not necessarily (3.22)), and consider the problem

$$\mathcal{B}p_{\beta,k}(x,t) = \mathcal{L}p_{\beta,k}(x,t) \quad , \tag{4.1}$$

$$\frac{\partial^i p_{\beta,k}(x,0)}{\partial t^i} = \begin{cases} 0 & , 0 \le i < \ell \quad , \quad i \ne k \, , \\ x^\beta & , i = k \, . \end{cases} \tag{4.2}$$

According to Theorem 1, there exists the polynomial solution

$$p_{\beta,k}(x,t) = \frac{(\nu+k/\ell)!}{k!} \sum_{m=0}^{\infty} \frac{\mathcal{L}^m\left(x^\beta\right)\, t^{m\ell+k}}{\ell^{m\ell}(\nu+m+k/\ell)!} \quad . \tag{4.3}$$

We introduce some notation in order to write (4.3) in terms of the roots $\{-\ell\nu_i\}$ of the auxiliary polynomial B of $\mathcal{B}$. Let I denote the number of indices in the summation (3.2) corresponding to nonzero a_α, so that we may label these indices as α^1, α^2, …, α^I, and the corresponding coefficients $\{a_\alpha\}$ as a_1, a_2, …, a_I. Then $\mathcal{L}$ can be written alternatively as

$$\mathcal{L} = \sum_{k=1}^{I} a_k \, \partial_x^{\alpha^k} \quad . \tag{4.4}$$

We recall the general multinomial formula

$$(c_1 + c_2 + \cdots + c_I)^m = \sum_{|\sigma|=m} \frac{m!}{\sigma!} \, c^\sigma \quad , \tag{4.5}$$

where $c = (c_1, c_2, \ldots, c_I)$, $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_I)$, and the summation is over all multi-indices σ in $\mathbb{R}^I$. We define a "vector of multi-indices" $\widehat{\alpha}$, and vector of coefficients a, according to

$$\widehat{\alpha} = \left(\alpha^1, \alpha^2, \cdots, \alpha^I\right) \qquad , \qquad a = (a_1, a_2, \cdots, a_I) \quad , \tag{4.6}$$

and introduce a "dot product"

$$\widehat{\alpha} \cdot \sigma = \sigma_1 \alpha^1 + \sigma_2 \alpha^2 + \cdots + \sigma_I \alpha^I \quad .$$

With this notation, we may write

$$\mathcal{L}^m = \left(a_1 \partial_x^{\alpha^1} + a_2 \partial_x^{\alpha^2} + \cdots + a_I \partial_x^{\alpha^I}\right)^m = \sum_{|\sigma|=m} \frac{m!}{\sigma!} \, a^\sigma \, \partial_x^{\widehat{\alpha}\cdot\sigma} \quad , \tag{4.7}$$

so that (4.3) becomes

$$\begin{aligned} p_{\beta,k}(x,t) &= \frac{(\nu + k/\ell)!}{k!} \sum_{m=0}^{\infty} \frac{t^{m\ell+k}}{\ell^{m\ell} (\nu + m + k/\ell)!} \sum_{|\sigma|=m} \frac{m!}{\sigma!} \, a^\sigma \, \partial_x^{\widehat{\alpha}\cdot\sigma} \left(x^\beta\right) \\ &= \frac{(\nu + k/\ell)!}{k!} \sum_{\sigma} \frac{t^{k+\ell|\sigma|}}{\ell^{\ell|\sigma|} (\nu + |\sigma| + k/\ell)!} \frac{|\sigma|!}{\sigma!} \, a^\sigma \, \partial_x^{\widehat{\alpha}\cdot\sigma} \left(x^\beta\right) \quad , \end{aligned}$$

with the summation over all multi-indices σ in $\mathbb{R}^I$. Using the formula

$$\partial^\gamma \left(x^\beta\right) = \begin{cases} x^{\beta-\gamma} \beta! / (\beta - \gamma)! & \text{, if } \gamma \leq \beta, \\ 0 & \text{, otherwise,} \end{cases} \tag{4.8}$$

where $\gamma \leq \beta$ means $\gamma_k \leq \beta_k$ for each k, we may finally write our polynomial solution of problem (4.1) – (4.2) as

$$p_{\beta,k}(x,t) = \frac{\beta! \, (\nu + k/\ell)!}{k!} \sum_{\widehat{\alpha}\cdot\sigma \leq \beta} \frac{|\sigma|! \, a^\sigma \, x^{\beta - \widehat{\alpha}\cdot\sigma} \, t^{k+\ell|\sigma|}}{\sigma! \, \ell^{\ell|\sigma|} \, (\nu + |\sigma| + k/\ell)! \, (\beta - \widehat{\alpha}\cdot\sigma)!} \quad , \tag{4.9}$$

with the summation over all multi-indices σ in $\mathbb{R}^I$ such that $\widehat{\alpha}\cdot\sigma \leq \beta$. (Since (3.2) has no zero order term, $\widehat{\alpha}\cdot\sigma \leq \beta$ is possible for only a finite number of multi-indices σ.)

For γ a multi-index in $\mathbb{R}^n$, we apply the operator ∂_x^γ to both sides of (4.3) and obtain

$$\partial_x^\gamma p_{\beta,k}(x,t) = \frac{(\nu+k/\ell)!}{k!}\sum_{m=0}^{\infty}\frac{\mathcal{L}^m\left(\partial^\gamma x^\beta\right)\ t^{m\ell+k}}{\ell^{m\ell}(\nu+m+k/\ell)!} \quad .$$

In view of (4.8) this equation leads to

$$\partial_x^\gamma p_{\beta,k}(x,t) = \begin{cases} \frac{\beta!}{(\beta-\gamma)!}p_{\beta-\gamma,k}(x,t) & \text{, if } \gamma\le\beta, \\ 0 & \text{, otherwise.}\end{cases} \tag{4.10}$$

Example 3. *Consider again the special case considered in Example 2,*

$$\mathcal{B} = \frac{\partial^\ell}{\partial t^\ell} \quad ,$$

when the roots $\{-\nu\ell\}$ *of the auxiliary polynomial are* $\{0,1,\ldots,\ell-1\}$*. We may take*

$$\nu = \{0,-1/\ell,-2/\ell,\cdots,-(\ell-1)/\ell\} \quad .$$

A calculation gives the simplification

$$\frac{(\nu+k/\ell)!}{\ell^{\ell|\sigma|}(\nu+|\sigma|+k/\ell)!} = \frac{k!}{(k+\ell|\sigma|)!} \quad ;$$

consequently, (4.9) simplifes to

$$p_{\beta,k}(x,t) = \beta!\sum_{\widehat{\alpha}\cdot\sigma\le\beta}\frac{|\sigma|!}{\sigma!}\frac{a^\sigma\ x^{\beta-\widehat{\alpha}\cdot\sigma}\ t^{k+\ell|\sigma|}}{(\beta-\widehat{\alpha}\cdot\sigma)\ \ (k+\ell|\sigma|)!} \quad .$$

This formula was derived in the paper [18] *of the present authors. The case* $\ell = 1$ *is discussed in some detail in the authors' papers* [15, 16, 17]*, where pointwise bounds are established on the polynomials, and series expansions in the polynomials are investigated.*

Example 4. *Next consider the equation*

$$\frac{\partial^2 u(x,t)}{\partial t^2} + \frac{2b+1}{t}\frac{\partial u(x,t)}{\partial t} = \varepsilon\Delta u(x,t) = \varepsilon\sum_{i-1}^{n}\frac{\partial^2 u(x,t)}{\partial {x_i}^2} \quad , \tag{4.11}$$

where $b,\varepsilon\in\mathbb{C}$ *and* $\varepsilon\neq 0$*. When* $\varepsilon = 1$ *the equation is the "Euler-Poisson-Darboux" equation, and when* $\varepsilon = -1$ *it is the "Beltrami equation". On the left we have a Bessel operator of order* $\ell = 2$*, with* $b_0 = 0$ *and auxiliary polynomial*

$$B(z) = z(z+2b) \quad ,$$

and hence with $\nu = (0,b)$*. On the right is an operator* $\mathcal{L}$*, for which the vectors* $\widehat{\alpha}$ *and* a *of (4.6) become the n-vectors*

$$\widehat{\alpha} = (2e_1, 2e_2, \ldots, 2e_n) \qquad , \qquad a = (\varepsilon,\ldots,\varepsilon) \quad ,$$

where e_i *is the i-th unit multi-index in* $\mathbb{R}^n$*. We assume that b is not a negative integer, so that (3.19) holds with* $k = 0$*. Then, for any multi-index*

β in $\mathbb{R}^n$, there is a polynomial solution p_β of (4.11) satisfying also the initial condition

$$p_\beta(x,0) = x^\beta \qquad , \qquad \frac{\partial}{\partial t} p_\beta(x,0) = 0 \quad .$$

Observing that, for σ any multi-index in $\mathbb{R}^n$,

$$\widehat{\alpha} \cdot \sigma = 2\sigma_1 e_1 + 2\sigma_2 e_2 + \cdots + 2\sigma_n e_n = 2\sigma \quad ,$$

we may write the solution (4.9) (with $k = 0$) as

$$p_\beta(x,t) = \beta! b! \sum_{2\sigma \le \beta} \frac{\varepsilon^{|\sigma|} \; x^{\beta - 2\sigma} \; t^{2|\sigma|}}{4^{|\sigma|} \; \sigma! \; (b + |\sigma|)! \; (\beta - 2\sigma)!} \quad ,$$

where the sum is over all multi-indices σ in $\mathbb{R}^n$ such that $2\sigma \le \beta$. For the cases $\varepsilon = 1$ and $\varepsilon = -1$ these polynomials were first discussed by E. P. Miles and E. Williams [22]*, and later in more detail by Bragg and Dettman* [3, 5]*.*

5. The Function E

Representations (3.16) and (3.18) motivate the introduction of a function E as the formal sum

$$E(t,\tau;\nu,k) = \frac{t^k}{k!} + \frac{1}{k!} \sum_{m=1}^{\infty} \frac{t^{m\ell + k} \tau^m}{\prod_{j=1}^m B(k + j\ell)} \tag{5.1}$$

$$= \frac{(\nu + k/\ell)!}{k!} \sum_{m=0}^{\infty} \frac{t^{m\ell + k} \tau^m}{\ell^{m\ell} (\nu + m + k/\ell)!} \quad . \tag{5.2}$$

We view t and τ ($\tau \in \mathbb{Z}$) as variables of this function, and ν and k as parameters. Then both (3.16) and (3.18) can be written symbolically with the brief notation

$$u(x,t) = E(t,\mathcal{L};\nu,k)\, q(x) \quad . \tag{5.3}$$

We discuss properties of the function E. As in Section 3, we assume that

$$B(k) = 0 \qquad , \qquad B(k + j\ell) \neq 0 \quad , \quad j = 1, 2, 3, \ldots \quad .$$

Equivalently, in terms of the roots $\{-\ell\nu_i : i = 1, 2, \ldots, \ell\}$ of B, we assume $\nu_i = -k/\ell$ for some i, and that no value $\nu_i + k/\ell$ is a negative integer. It follows that in (5.2) the factorials

$$(\nu + m + k/\ell)! = \prod_{i=1}^{\ell} (\nu_i + m + k/\ell)!$$

are indeed defined. With use of the ratio test it is easily deduced that the power series (5.2) converges uniformly for t and τ on compact subsets of $\mathbb{C}$, representing a C^∞ function with respect to the variables t and τ, and that termwise differentiation of all orders with respect to t and/or τ is valid.

Termwise differentiation of version (5.1) of E confirms that

$$\frac{\partial^i E(0,\tau;\nu,k)}{\partial t^i} = \begin{cases} 1 & \text{, if } i = k, \\ 0 & \text{, if } i \neq k,\ 0 \le i < \ell. \end{cases} \tag{5.4}$$

Applying the operator $\mathcal{B}$ to (5.1) gives, with use of of (2.4) and $B(k)=0$,

$$\begin{aligned}\mathcal{B}E(t,\tau;\nu,k) &= \frac{B(k)\,t^{k-\ell}}{k!}+\frac{1}{k!}\sum_{m=1}^{\infty}\frac{B(k+m\ell)\,t^{m\ell+k-\ell}\tau^m}{\prod_{j=1}^{m}B(k+j\ell)}\\ &= 0+\frac{1}{k!}\frac{B(k+\ell)\,t^k\tau}{\prod_{j=1}^{1}B(k+j\ell)}+\frac{1}{k!}\sum_{m=2}^{\infty}\frac{B(k+m\ell)\,t^{m\ell+k-\ell}\tau^m}{\prod_{j=1}^{m}B(k+j\ell)}\\ &= \tau\,\frac{t^k}{k!}+\frac{1}{k!}\sum_{m=1}^{\infty}\frac{t^{m\ell+k}\tau^{m+1}}{\prod_{j=1}^{m}B(k+j\ell)}\quad ,\end{aligned}$$

and hence

$$\mathcal{B}E(t,\tau;\nu,k)=\tau E(t,\tau;\nu,k)\quad . \tag{5.5}$$

6. Generating Functions

The polynomials $\{p_{\beta,k}\}$ of Section 4 can be obtained also through a generating function. As in Section 5 we continue to presume the conditions

$$B(k)=0\qquad,\qquad B(k+j\ell)\neq 0\quad,\quad j=1,2,3,\ldots\quad ,$$

where k is a fixed integer, $0\leq k<\ell$, thereby ensuring the existence of the function E of (5.1) – (5.2). We associate with the operator $\mathcal{L}$ of (3.2) a polynomial

$$Q(y)=\sum_{\alpha}a_\alpha y^\alpha=\sum_{i=1}^{I}a_i y^{\alpha^i}\quad ,$$

where $y=(y_1,\ldots,y_n)$ is a vector in the same space ($\mathbb{R}^n$ or $\mathbb{C}^n$) as x. Finally we define

$$G_k(x,t,y)=e^{x\cdot\overline{y}}E(t,Q(y);\nu,k)\quad , \tag{6.1}$$

where $x\cdot y=x_1\overline{y_1}+\cdots+x_n\overline{y_n}$ is the usual dot product, $\overline{y}=(\overline{y_1},\cdots,\overline{y_n})$, and

$$e^{x\cdot\overline{y}}=e^{x_1y_1+\cdots+x_ny_n}=\sum_{\gamma}\frac{x^\gamma y^\gamma}{\gamma!}\quad , \tag{6.2}$$

with the sum over all multi-indices γ in $\mathbb{R}^n$.

Application of (4.5) gives

$$\begin{aligned}Q(y)^m=\left(\sum_{i=1}^{I}a_i y^{\alpha^i}\right)^m &= \sum_{|\sigma|=m}\frac{m!}{\sigma!}\left(a_1y^{\alpha^1},a_2y^{\alpha^2},\ldots,a_ny^{\alpha^n}\right)^\sigma\\ &= \sum_{|\sigma|=m}\frac{m!}{\sigma!}a^\sigma y^{\widehat{\alpha}\cdot\sigma}\quad ,\end{aligned}$$

where the last sum is over all multi-indices σ in $\mathbb{R}^I$. With these formulas and (5.2) we may write

$$\begin{aligned} G_k(x,t,y) &= e^{x\cdot y}\frac{(\nu+k/\ell)!}{k!}\sum_{m=0}^{\infty}\frac{t^{m\ell+k}Q(y)^m}{\ell^{m\ell}(\nu+m+k/\ell)!} \\ &= \frac{(\nu+k/\ell)!}{k!}\sum_{\gamma}\sum_{m=0}^{\infty}\sum_{|\sigma|=m}\frac{x^\gamma y^\gamma}{\gamma!}\frac{t^{m\ell+k}}{\ell^{m\ell}(\nu+m+k/\ell)!}\frac{m!}{\sigma!}a^\sigma y^{\widehat{\alpha}\cdot\sigma} \\ &= \frac{(\nu+k/\ell)!}{k!}\sum_{\gamma}\sum_{\sigma}\frac{x^\gamma}{\gamma!}\frac{t^{k+|\sigma|\ell}}{\ell^{\ell|\sigma|}(\nu+|\sigma|+k/\ell)!}\frac{|\sigma|!}{\sigma!}a^\sigma y^{\gamma+\widehat{\alpha}\cdot\sigma} \quad . \end{aligned}$$

Straightforward application of the ratio test confirms that this last double series converges absolutely and uniformly for values of x, y and t in compact sets. The same statement is valid for all derivatives, mixed or unmixed, of the series with respect to the variables x, y, and t. It follows that G_k is an analytic function of the variables (x,y,t), the series can be differentiated termwise with respect to any combinations of the variables as often as desired, and the series and all differentiated series may be summed in any order. In particular, we may interchange orders of summation, first summing outside over powers y^β, as $\beta=\gamma+\widehat{\alpha}\cdot\sigma$ ranges over all multi-indices in $\mathbb{R}^n$, getting

$$\begin{aligned} G_k(x,t,y) &= \frac{(\nu+k/\ell)!}{k!}\sum_{\beta}y^\beta\sum_{\gamma+\widehat{\alpha}\cdot\sigma=\beta}\frac{|\sigma|!}{\sigma!}\frac{a^\sigma x^\gamma t^{k+|\sigma|\ell}}{\ell^{\ell|\sigma|}\gamma!(\nu+|\sigma|+k/\ell)!} \\ &= \frac{(\nu+k/\ell)!}{k!}\sum_{\beta}y^\beta\sum_{\widehat{\alpha}\cdot\sigma\le\beta}\frac{|\sigma|!}{\sigma!}\frac{a^\sigma x^{\beta-\widehat{\alpha}\cdot\sigma}t^{k+|\sigma|\ell}}{\ell^{\ell|\sigma|}(\beta-\widehat{\alpha}\cdot\sigma)!(\nu+|\sigma|+k/\ell)!} \quad . \end{aligned}$$

Comparing the last equation with (4.9), we find that

$$G_k(x,t,y)=\sum_{\beta}p_{\beta,k}(x,t)\frac{y^\beta}{\beta!} \quad . \tag{6.3}$$

From (6.1), (5.5), (3.2) and (6.2) it follows that

$$\mathcal{B}G_k(x,t,y)=Q(y)\,G_k(x,t,y)=\mathcal{L}G_k(x,t,y) \quad , \tag{6.4}$$

and from (6.1) and (5.4) that

$$\frac{\partial^i G_k(x,0,y)}{\partial t^i}=\begin{cases} e^{x\cdot\overline{y}} & \text{, if } i=k, \\ 0 & \text{, if } i\neq k,\ 0\le i<\ell. \end{cases} \tag{6.5}$$

Then (6.3) and (6.4) yield

$$\sum_{\beta}\mathcal{B}p_{\beta,k}(x,t)\frac{y^\beta}{\beta!}=\sum_{\beta}\mathcal{L}p_{\beta,k}(x,t)\frac{y^\beta}{\beta!} \quad ,$$

and hence another proof of (4.1). Also, (6.3) implies

$$\frac{\partial^i G_k(x,0,y)}{\partial t^i} = \sum_\beta \frac{\partial^i p_{\beta,k}(x,0)}{\partial t^i}\frac{y^\beta}{\beta!} \quad ,$$

and then (6.5) and (6.2) (with γ replaced by β) produce another proof of (4.2). Consequently, the power series representation (6.3) could be used as a definition of the polynomials $\{p_{\beta,k}\}$.

7. Bessel Space Operator

As explained in Remark 1, Theorem 1 can be applied to other operators besides (3.2). We consider the special case when x is a scalar variable (either real or complex), and (3.2) is replaced with the s-th order Bessel operator $(s \geq 1)$

$$\mathcal{C} = \sum_{i=0}^{s} \frac{c_i}{t^{s-i}}\frac{\partial^i}{\partial x^i} \quad . \tag{7.1}$$

We assume $c_s \neq 0$, but not necessarily that $c_s = 1$. The auxiliary polynomial for $\mathcal{C}$ is

$$C(z) = c_0 + \sum_{i=1}^{s} c_i z(z-1)(z-2)\cdots(z-i+1) = c_s \prod_{i=1}^{s}(z+s\mu_i) \quad , \tag{7.2}$$

with roots $\{-s\mu_i : 1 \leq i \leq s\}$, repeated according to multiplicity. The operator $\mathcal{C}$ factors into

$$\mathcal{C} = \frac{c_s}{t^s}\prod_{i=1}^{s}\left(t\frac{\partial}{\partial t} + s\mu_i\right) \quad ,$$

and for any integer j,

$$\mathcal{C}\left(x^j\right) = c_s x^{j-s}\prod_{i=1}^{s}(j+s\mu_i) = C(j)\,x^{j-s} \quad . \tag{7.3}$$

Let $\mathcal{B}$ again be the Bessel operator (2.1). We seek a polynomial solution of the Cauchy problem

$$\mathcal{B}u(x,t) = \mathcal{C}u(x,t) \quad , \tag{7.4}$$

$$\frac{\partial^i u(x,0)}{\partial t^i} = \begin{cases} 0 & ,\ 0 \leq i < \ell \quad , \quad i \neq k\ , \\ x^\beta & ,\ i = k\ , \end{cases} \tag{7.5}$$

where β is a nonnegative integer. We assume again condition (3.19) of Theorem 1. Then formulas (3.16) and (3.18), with $\mathcal{L}$ replaced by $\mathcal{C}$ and $q(x)$ by x^β, give potential solutions of problem (7.4) – (7.5). However, in view of Remark 1, we must be certain that each $\mathcal{C}^m\left(x^\beta\right)$ is a polynomial, and that $\mathcal{C}^m\left(x^\beta\right)=0$ for m sufficiently large. By iteration of (7.3),

$$\mathcal{C}^m\left(x^\beta\right)=x^{\beta-ms}\prod_{j=0}^{m-1}C\left(\beta-js\right)\quad,\quad m\geq 1\quad. \tag{7.6}$$

The expression on the right will eventually vanish if $C\left(\beta-m_\beta s\right)=0$ for some nonnegative integer m_β. In this event the sums on the right of (3.16) and (3.18) terminate at $m=m_\beta$. In view of (7.6), to ensure that $\mathcal{C}^m\left(x^\beta\right)$ is a polynomial we require that $\beta-m_\beta s\geq 0$, or $m_\beta\leq\beta/s$.

For complex numbers z and w we use the notation

$$\binom{z}{w}=\frac{z!}{w!\left(z-w\right)!}$$

whenever the three factorials on the right are defined. For complex n-vectors u and v we write

$$\binom{u}{v}=\frac{u!}{v!\left(u-v\right)!}=\frac{\prod_{i=1}^n u_i!}{\prod_{i=1}^n v_i!\prod_{i=1}^n\left(u_i-v_i\right)!}=\prod_{i=1}^n\binom{u_i}{v_i}\quad,$$

again when all the required factorials are defined. For nonnegative integers m and for $z\in\mathbb{C}$ we define

$$[z]_0=1\quad,\quad[z]_m=z\left(z-1\right)\left(z-2\right)\cdots\left(z-m+1\right)\qquad(m\geq 1)\quad.$$

For n-vectors we prescribe

$$[u]_m=[u_1]_m\,[u_2]_m\cdots[u_n]_m=\prod_{i=1}^n[u_i]_m\quad.$$

We have verified most of the following :

Theorem 2. *Let $\mathcal{B}$ be the Bessel operator (2.1), with auxiliary polynomial (2.2), and let $\mathcal{C}$ be the Bessel operator (7.1), with auxiliary polynomial (7.2). Let k be an integer, $0\leq k<\ell$, and suppose that*

$$B\left(k\right)=0\quad,\quad B\left(k+j\ell\right)\neq 0\quad,\quad j=1,2,3,\ldots\quad. \tag{7.7}$$

Let β be a nonnegative integer such that, for some integer m_β,

$$0\leq m_\beta\leq\beta/s\quad,\quad C\left(\beta-m_\beta s\right)=0\quad. \tag{7.8}$$

Then the Cauchy problem (7.4) – (7.5) has a polynomial solution $u = p_{\beta,k}$*, with the explicit representations*

$$p_{\beta,k}(x,t) = x^\beta \frac{t^k}{k!} + \frac{1}{k!}\sum_{m=1}^{m_\beta} \frac{\mathcal{C}^m\left(x^\beta\right) t^{m\ell+k}}{\prod_{j=1}^m B(k+j\ell)} \tag{7.9}$$

$$= \frac{(\nu+k/\ell)!}{k!}\sum_{m=0}^{m_\beta} \frac{\mathcal{C}^m\left(x^\beta\right) t^{m\ell+k}}{\ell^{m\ell}(\nu+m+k/\ell)!} \tag{7.10}$$

$$= \frac{(\nu+k/\ell)!}{k!}\sum_{m=0}^{m_\beta} \frac{[\mu+\beta/s]_m {c_s}^m s^{ms} x^{\beta-ms} t^{m\ell+k}}{\ell^{m\ell}(\nu+m+k/\ell)!} \quad . \tag{7.11}$$

If

$$j \in \mathbb{Z},\ j > 0 \Longrightarrow C(\beta - m_\beta s + js) \neq 0 \quad , \tag{7.12}$$

then these formulas may be written as

$$p_{\beta,k}(x,t) = \frac{(\nu+k/\ell)!\,(\mu+\beta/s)!}{k!}\sum_{m=0}^{m_\beta} \frac{{c_s}^m s^{ms} x^{\beta-ms} t^{m\ell+k}}{\ell^{m\ell}(\nu+m+k/\ell)!\,(\mu+\beta/s-m)!} \quad . \tag{7.13}$$

If

$$B(j) \neq 0 \quad , \quad j = \ell,\ \ell+1,\ \ell+2,\ \ldots \quad , \tag{7.14}$$

then there is only one polynomial solution of the problem (7.4) - (7.5).

Proof. The preceding discussion has verified all assertions, except formula (7.11), and formula (7.13) under the additional assumption (7.12).

For $1 \leq m \leq m_\beta$ in (7.10), we may use (7.6) and (7.2) to write

$$\mathcal{C}^m\left(x^\beta\right) = x^{\beta-ms}\prod_{j=0}^{m-1} c_s \prod_{i=1}^{s}(\beta - js + s\mu_i)$$

$$= {c_s}^m s^{ms} x^{\beta-ms} \prod_{i=1}^{s}\prod_{j=0}^{m-1}(\mu_i + \beta/s - j) \tag{7.15}$$

$$= {c_s}^m s^{ms} x^{\beta-ms} \prod_{i=1}^{s}[\mu_i + \beta/s]_m$$

$$= {c_s}^m s^{ms} x^{\beta-ms}[\mu+\beta/s]_m \quad . \tag{7.16}$$

As this formula holds also for $m = 0$, we substitute it into (7.10) to obtain (7.11).

Next assume (7.12) holds. Given any negative integer N we have $m_\beta - m - N > 0$, and hence by (7.12),

$$C(\beta - m_\beta s + (m_\beta - m - N)s) = C(\beta - ms - Ns) \neq 0 \quad .$$

Therefore, for $i = 1, 2, \ldots, s$,

$$\beta - ms - Ns \neq -s\mu_i \quad ,$$
$$\mu_i + \beta/s - m \neq N \quad .$$

Thus we may apply (3.17) in (7.15) to write

$$\mathcal{C}^m \left(x^\beta\right) = {c_s}^m s^{ms} x^{\beta - ms} \prod_{i=1}^{s} \frac{(\mu_i + \beta/s)!}{(\mu_i + \beta/s - m)!}$$
$$= {c_s}^m s^{ms} x^{\beta - ms} \frac{(\mu + \beta/s)!}{(\mu + \beta/s - m)!} \quad .$$

Substitution into (7.10) gives (7.13). □

Example 5. *Suppose the Bessel operator $\mathcal{C}$ has no zero order term, so that $c_0 = 0$ in (7.2) and $C(0) = 0$. If $\beta = Ms$ for some nonnegative integer M, then (7.8) is satisfied with $m_\beta = M$. If we assume further that $\mathcal{B}$ has no zero order term, then (7.7) is satisfied with $k = 0$ provided that no root of B is a positive integral multiple of ℓ. Then the Cauchy problem*

$$\mathcal{B}u(x,t) = \mathcal{C}u(x,t) \quad ,$$
$$\frac{\partial^i u(x,0)}{\partial t^i} = \begin{cases} x^{Ms} & , i = 0 \quad , \\ 0 & , 1 \leq i < \ell \quad , \end{cases}$$

has a solution

$$p_{Ms,0}(x,t) = x^{Ms} + \sum_{m=1}^{M} \frac{\mathcal{C}^m \left(x^{Ms}\right) t^{m\ell}}{\prod_{j=1}^{m} B(j\ell)} \quad .$$

Condition (7.12) holds if also no root of C is a positive integral multiple of s, in which event

$$p_{Ms,0}(x,t) = \nu! \, (\mu + M)! \sum_{m=0}^{M} \frac{{c_s}^m \; s^{ms} x^{(M-m)s} t^{m\ell}}{\ell^{m\ell} (\nu + m)! \, (\mu + M - m)!} \quad .$$

Example 6. *Consider the Cauchy problem*

$$\frac{\partial}{\partial t} u(r,t) = \frac{\partial^2}{\partial r^2} u(r,t) + \frac{2c+1}{r} \frac{\partial}{\partial r} u(r,t) \quad , \tag{7.17}$$

$$u(r,0) = r^\beta \quad , \tag{7.18}$$

where $c \in \mathbb{C}$. When $2c + 1 = n - 1$, the right side of (7.17) is the n-dimensional Laplacian in radial coordinates. We have $\ell = 1$, $s = 2$, $b_\ell = c_s = 1$, and

$$\mathcal{B} = \frac{\partial}{\partial t} \quad , \quad B(z) = z \quad , \quad \nu = 0 \quad ,$$

$$\mathcal{C} = \frac{\partial^2}{\partial r^2} + \frac{2c+1}{r} \frac{\partial}{\partial r} \quad , \quad C(z) = z(z + 2c) \quad , \quad \mu = (0, c) \quad .$$

Then (7.7) holds with $k = 0$, while (7.8) will hold if we choose $\beta = 2M$ for some nonnegative integer M, in which case $m_\beta = M$. As (7.14) likewise is valid, formula (7.10) gives the unique polynomial solution p_β of (7.17) – (7.18) as

$$p_{2M}(r,t) = \sum_{m=0}^{M} \frac{\mathcal{C}^m\left(r^{2M}\right)t^m}{m!} \quad .$$

If we assume c is not a negative integer, then (7.12) holds, and from (7.13) we have the alternate representation

$$p_{2M}(r,t) = (\mu + M)! \sum_{m=0}^{M} \frac{2^{2m} r^{2(M-m)} t^m}{m!\,(\mu + M - m)!} \quad .$$

Substituting $\mu + M = (M, M + c)$, $\mu + M - m = (M - m, M - m + c)$ gives

$$\begin{aligned} p_{2M}(r,t) &= (M+c)! \sum_{m=0}^{M} \binom{M}{m} \frac{r^{2(M-m)}(4t)^m}{(M-m+c)!} \\ &= (M+c)! \sum_{m=0}^{M} \binom{M}{m} \frac{r^{2m}(4t)^{M-m}}{(m+c)!} \quad . \end{aligned}$$

The polynomials $\{p_{2M}\}$ are called "radial heat polynomials". They have been studied in some detail by Bragg [2] *and Haimo* [8, 9, 10, 12, 13, 14].

Example 7. *Consider the partial differential equation*

$$\frac{\partial^2}{\partial t^2} u(r,t) = \varepsilon \left[\frac{\partial^2}{\partial r^2} u(r,t) + \frac{2c+1}{r} \frac{\partial}{\partial r} u(r,t) \right] \quad , \tag{7.19}$$

where $c, \varepsilon \in \mathbb{C}$ and $\varepsilon \neq 0$. When $\varepsilon = 1$ and $2c + 1 = n - 1$, the equation is called the "radial wave equation", and when $\varepsilon = -1$ and $2c+1 = n-1$ it is the "radial Laplace equation". We have $\ell = s = 2$, $b_\ell = 1$, $c_s = \varepsilon$, and

$$\mathcal{B} = \frac{\partial^2}{\partial t^2} \quad , \quad B(z) = z(z-1) \quad , \quad \nu = (0, -1/2) \quad ,$$

$$\mathcal{C} = \varepsilon \frac{\partial^2}{\partial r^2} + \frac{\varepsilon(2c+1)}{r} \frac{\partial}{\partial r} \quad , \quad C(z) = \varepsilon z (z + 2c) \quad , \quad \mu = (0, c) \quad .$$

Then (7.7) holds with both $k = 0$ and $k = 1$, (7.8) holds if we choose $\beta = 2M$ for some nonnegative integer M, with $m_\beta = M$. Also, (7.14) is valid. By Theorem 2, we have unique polynomial solutions of (7.19) satisfying either of the Cauchy conditions

$$u(r,0) = r^{2M} \quad , \quad \frac{\partial u(r,0)}{\partial t} = 0 \quad , \tag{7.20}$$

$$u(r,0) = 0 \quad , \quad \frac{\partial u(r,0)}{\partial t} = r^{2M} \quad . \tag{7.21}$$

Formula (7.10) gives for problem (7.19), (7.20) the solution

$$p_{2M,0}(r,t) = \nu! \sum_{m=0}^{M} \frac{\mathcal{C}^m \left(r^{2M}\right) t^{2m}}{2^{2m} (\nu+m)!} \quad ,$$

and for problem (7.19), (7.21),

$$p_{2M,1}(r,t) = (\nu+1/2)! \sum_{m=0}^{M} \frac{\mathcal{C}^m \left(r^{2M}\right) t^{2m+1}}{2^{2m} (\nu+m+1/2)!} \quad .$$

If c is not a negative integer then (7.12) holds, and (7.13) gives the alternate representations

$$\begin{aligned} p_{2M,0}(r,t) &= \nu! \, (\mu+M)! \sum_{m=0}^{M} \frac{\varepsilon^m r^{2(M-m)} t^{2m}}{(\nu+m)! \, (\mu+M-m)!} \\ &= (-1/2)! \, (c+M)! \sum_{m=0}^{M} \binom{M}{m} \frac{\varepsilon^m r^{2(M-m)} t^{2m}}{(m-1/2)! \, (c+M-m)!} \quad , \end{aligned}$$

$$\begin{aligned} p_{2M,1}(r,t) &= (\nu+1/2)! \, (\mu+M)! \sum_{m=0}^{M} \frac{\varepsilon^m r^{2(M-m)} t^{2m+1}}{(\nu+m+1/2)! \, (\mu+M-m)!} \\ &= (1/2)! \, (c+M)! \sum_{m=0}^{M} \binom{M}{m} \frac{\varepsilon^m r^{2(M-m)} t^{2m+1}}{(m+1/2)! \, (c+M-m)!} \quad . \end{aligned}$$

Use of (3.17) leads to

$$(m-1/2)! = (-1/2)! \prod_{j=0}^{m-1} (m-1/2-j) = (-1/2)! \frac{(2m)!}{2^{2m} m!} \quad ,$$

$$(m+1/2)! = (1/2)! \prod_{j=0}^{m-1} (m+1/2-j) = (1/2)! \frac{(2m+1)!}{2^{2m} m!} \quad .$$

With use of these identities we may simplify the above representations to

$$p_{2M,0}(r,t) = \sum_{m=0}^{M} \frac{\binom{M}{m}\binom{M+c}{m}}{\binom{2m}{m}} \varepsilon^m r^{2(M-m)} (2t)^{2m} \quad ,$$

$$p_{2M,1}(r,t) = \sum_{m=0}^{M} \frac{\binom{M}{m}\binom{M+c}{m}}{\binom{2m}{m} 2 (2m+1)} \varepsilon^m r^{2(M-m)} (2t)^{2m+1} \quad .$$

Observe that

$$p_{2M,0}(r,t) = \frac{\partial}{\partial t} p_{2M,1}(r,t) \quad .$$

In the cases $\varepsilon = 1$ and $\varepsilon = -1$ these polynomials have been investigated by Bragg and Dettman [5].

Example 8. *The "radial Euler-Poisson-Darboux equation" and "radial Beltrami equation" are special cases of*

$$\frac{\partial^2 u\,(r,t)}{\partial t^2} + \frac{2b+1}{t}\frac{\partial u\,(r,t)}{\partial t} = \varepsilon \left[\frac{\partial^2 u\,(r,t)}{\partial r^2} + \frac{2c+1}{t}\frac{\partial u\,(r,t)}{\partial r}\right] \quad , \tag{7.22}$$

obtained by taking $\varepsilon = +1$ *and* $\varepsilon = -1$*, respectively. Here* $b, c, \varepsilon \in \mathbb{C}$*, and* $\varepsilon \neq 0$*. For the Bessel operator* $\mathcal{B}$ *on the left we have*

$$B\,(z) = z\,(z + 2b) \qquad , \qquad \nu = (0, b) \quad ,$$

and for the Bessel operator $\mathcal{C}$ *on the right,*

$$C\,(z) = \varepsilon z\,(z + 2c) \qquad , \qquad \mu = (0, c) \quad .$$

We assume b *is not a negative integer, so that (7.7) holds with* $k = 0$*. We choose* $\beta = 2M$ *for some nonnegative integer* M*, so that (7.8) holds with* $m_\beta = M$*. By Theorem 2, there exists a polynomial solution* p_β *of (7.22) satisfying the initial condition*

$$p_{2M}\,(r, 0) = r^{2M} \qquad , \qquad \frac{\partial}{\partial t} p_{2M}\,(r, 0) = 0 \quad .$$

Formula (7.10) gives

$$p_{2M}\,(r,t) = \nu! \sum_{m=0}^{M} \frac{\mathcal{C}^m\left(r^{2M}\right) t^{2m}}{2^{2m}\,(\nu + m)!} = b! \sum_{m=0}^{M} \frac{\mathcal{C}^m\left(r^{2M}\right) t^{2m}}{2^{2m} m!\,(b+m)!} \quad .$$

If we assume that also c *is not a negative integer, then (7.12) holds, and (7.13) gives*

$$\begin{aligned} p_{2M}\,(r,t) &= \nu!\,(\mu + M)! \sum_{m=0}^{M} \frac{\varepsilon^m r^{2(M-m)} t^{2m}}{(\nu+m)!\,(\mu + M - m)!} \\ &= b!\,(c+M)! \sum_{m=0}^{M} \binom{M}{m} \frac{\varepsilon^m r^{2(M-m)} t^{2m}}{(b+m)!\,(c+M-m)!} \quad . \end{aligned}$$

In the cases $\varepsilon = +1$ *and* $\varepsilon = -1$*, these polynomials were studied as well by Bragg and Dettman* [5].

8. More Bessel Space Operators

We extend the theory of the previous section to space operators in n variables. For each coordinate space variable x_i of $x = (x_1, \ldots, x_n)$, we introduce a Bessel operator $\mathcal{C}_i$, of order s_i in this variable,

$$\mathcal{C}_i = \sum_{j=0}^{s_i} \frac{c_{ij}}{{x_i}^{s_i - j}} \frac{\partial^j}{\partial {x_i}^j} \qquad (1 \leq i \leq n) \quad .$$

The coefficients $\{c_{ij}\}$ are complex numbers, and the leading coefficient, c_{is_i}, of each $\mathcal{C}_i$ is nonzero. The auxiliary polynomial of $\mathcal{C}_i$ is

$$C_i(z) = c_{i0} + \sum_{j=1}^{s_i} c_{ij} z(z-1)(z-2)\cdots(z-j+1) = c_{is_i} \prod_{j=1}^{s_i} (z + s_i \mu_{ij}) \quad ,$$

with the complex numbers $\{-s_i\mu_{ij} : j = 1, \ldots, s_i\}$ the roots of C_i. For any integer r,

$$\mathcal{C}_i(z^r) = c_{is_i} z^{r-s_i} \prod_{j=1}^{s_i} (r + s_i\mu_{ij}) = C_i(r)\, z^{r-s_i} \quad . \tag{8.1}$$

We designate vectors

$$\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_n) \quad , \quad \mu_i = (\mu_{i1}, \mu_{i2}, \cdots, \mu_{is_i}) \quad (1 \le i \le n) \quad ,$$

and given a multi-index α in $\mathbb{R}^n$ we define

$$\mathcal{C}^\alpha = \mathcal{C}_1{}^{\alpha_1} \mathcal{C}_2{}^{\alpha_2} \cdots \mathcal{C}_n{}^{\alpha_n} \quad .$$

Finally, we let $\mathcal{L}$ denote the operator

$$\mathcal{L} = \sum_\alpha a_\alpha \mathcal{C}^\alpha \quad ,$$

where the coefficients $\{a_\alpha\}$ are complex constants, the sum is over a finite collection of multi-indices α in $\mathbb{R}^n$, and $\mathcal{L}$ has no zero order term (corresponding to $\alpha = (0, \ldots, 0)$). We number these multi-indices and coefficients as in (4.6).

Let $\mathcal{B}$ denote our usual Bessel "time operator" (2.1), with auxiliary polynomial (2.2) and leading coefficient $b_\ell = 1$. Given a polynomial $q = q(x)$, we consider the Cauchy problem

$$\mathcal{B}u(x,t) = \mathcal{L}u(x,t) \quad , \tag{8.2}$$

$$\frac{\partial^i u(x,0)}{\partial t^i} = \begin{cases} 0 & , 0 \le i < \ell \quad , \quad i \ne k \\ q(x) & , i = k \quad . \end{cases} \tag{8.3}$$

As explained in Remark 1, Theorem 1 will apply to this problem, resulting in the polynomial solution

$$u(x,t) = q(x)\frac{t^k}{k!} + \frac{1}{k!}\sum_{m=1}^{\infty} \frac{\mathcal{L}^m q(x)\, t^{m\ell+k}}{\prod_{j=1}^m B(k+j\ell)} \tag{8.4}$$

$$= \frac{(\nu + k/\ell)!}{k!} \sum_{m=0}^{\infty} \frac{\mathcal{L}^m q(x)\, t^{m\ell+k}}{\ell^{m\ell}(\nu + m + k/\ell)!} \quad , \tag{8.5}$$

provided only that each power $\mathcal{L}^m q$ is a polynomial, vanishing identically for m sufficiently large.

We specialize to the case

$$q(x) = x^\beta = x_1{}^{\beta_1} x_2{}^{\beta_2} \cdots x_n{}^{\beta_n} \quad ,$$

with β a multi-index in $\mathbb{R}^n$, and give conditions under which indeed Theorem 1 applies. We assume $\mathcal{B}$ and each $\mathcal{C}_i$ has no zero order term, and that no root of B is a positive integral multiple of ℓ. Since (3.19) holds with $k = 0$, our Cauchy condition is

$$\frac{\partial^i u(x,0)}{\partial t^i} = \begin{cases} x^\beta & , i = 0 \\ 0 & , 1 \le i < \ell \end{cases} \quad . \tag{8.6}$$

We assume further that there are nonnegative integers $\{\gamma_i\}$ such that

$$\beta_i = \gamma_i s_i \qquad , \qquad 1 \le i \le n \quad .$$

Then $C_i(0) = 0$ for each i, and iteration of (8.1) shows that, for any positive integer J,

$$\mathcal{C}_i{}^J\left(x_i{}^{\beta_i}\right) = \mathcal{C}_i{}^J\left(x_i{}^{\gamma_i s_i}\right) = x_i{}^{(\gamma_i - J)s_i} \prod_{j=0}^{J-1} C_i(\gamma_i s_i - j s_i) \quad . \tag{8.7}$$

Thus, $J > \gamma_i$ implies $\mathcal{C}_i{}^J\left(x_i{}^{\beta_i}\right) = 0$. This argument shows that, for any multi-index α in $\mathbb{R}^n$, $\mathcal{C}^\alpha\left(x^\beta\right)$ is a multinomial and $\mathcal{C}^\alpha\left(x^\beta\right) = 0$ unless $\alpha \le \gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n)$. The analogues of (4.4) and (4.7) in this situation are

$$\mathcal{L} = \sum_{k=1}^{I} a_k \mathcal{C}^{\alpha^k} \qquad , \qquad \mathcal{L}^m = \sum_{|\sigma|=m} \frac{m!}{\sigma!} a^\sigma \mathcal{C}^{\widehat{\alpha}\cdot\sigma} \quad .$$

We substitute into (8.5), with $k = 0$, and obtain for the solution of problem (8.2), (8.6), with

$$\beta = (\gamma_1 s_1, \gamma_2 s_2, \cdots, \gamma_n s_n) \quad ,$$

the solution

$$p_\beta(x,t) = \nu! \sum_{m=0}^{\infty} \frac{\mathcal{L}^m\left(x^\beta\right) t^{m\ell}}{\ell^{m\ell}(\nu+m)!} \tag{8.8}$$

$$= \nu! \sum_{\widehat{\alpha}\cdot\sigma \le \gamma} \frac{t^{\ell|\sigma|}}{\ell^{\ell|\sigma|}(\nu+|\sigma|)!} \frac{|\sigma|!}{\sigma!} a^\sigma \mathcal{C}^{\widehat{\alpha}\cdot\sigma}\left(x^\beta\right) \quad . \tag{8.9}$$

The last sum is over multi-indices σ in $\mathbb{R}^I$. As $\mathcal{L}$ has no zero order term, this sum has only a finite number of terms, and p_β is a polynomial.

Example 9. *Bragg and Dettman* [4] *studied polynomial solutions of*

$$\left(\frac{\partial^2}{\partial t^2} + \frac{2b+1}{t}\frac{\partial}{\partial t}\right) u(x,t) = \sum_{i=1}^{n} \varepsilon_i \left(\frac{\partial^2}{\partial x_i{}^2} + \frac{2c_i+1}{x_i}\frac{\partial}{\partial x_i}\right) u(x,t), \tag{8.10}$$

$$u(x,0) = x^\beta \qquad , \qquad \frac{\partial u(x,0)}{\partial t} = 0 \quad , \tag{8.11}$$

where β has the special form

$$\beta = 2\gamma = (2\gamma_1, 2\gamma_2, \cdots, 2\gamma_n)$$

for some other multi-index γ. They required that each ε_i be either $+1$ or -1, but we stipulate only that each is a nonzero complex number. We assume b and each c_i are complex numbers, but that none of these quantities is a negative integer. For the operator on the left of (8.10) we have

$$\mathcal{B} = \frac{\partial^2}{\partial t^2} + \frac{2b+1}{t}\frac{\partial}{\partial t} \quad , \quad B(z) = z(z+2b) \quad , \quad \nu = (0,b) \quad .$$

The operator on the right of (8.10) can be written as

$$\mathcal{L} = \sum_{i=1}^{n} \varepsilon_i \mathcal{C}_i \quad , \tag{8.12}$$

where, for $1 \le i \le n$,

$$\mathcal{C}_i = \frac{\partial^2}{\partial {x_i}^2} + \frac{2c_i+1}{x_i}\frac{\partial}{\partial x_i} \; , \quad C_i(z) = z(z+2c_i) \; , \quad \mu_i = (0,c_i) \; . \tag{8.13}$$

As no root of B is a positive integral multiple of ℓ ($\ell = 2$), and each $\mathcal{C}_i$ has no zero order term, the preceding analysis applies. From (8.12) we see that the vector $\widehat{\alpha}$ of multi-indices and vector a of coefficients are

$$\widehat{\alpha} = (e_1, e_2, \cdots, e_n) \qquad , \qquad a = \varepsilon := (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) \quad , \tag{8.14}$$

where e_i denotes the i-th unit multi-index. For σ a multi-index in $\mathbb{R}^n$ we have $\widehat{\alpha} \cdot \sigma = \sigma$. We substitute these quantities into (8.9) and obtain as our polynomial solution of (8.10) – (8.11) the function

$$p_\beta(x,t) = \nu! \sum_{\sigma \le \gamma} \frac{t^{\ell|\sigma|}}{\ell^{\ell|\sigma|}(\nu+|\sigma|)!} \frac{|\sigma|!}{\sigma!} \varepsilon^\sigma \mathcal{C}^\sigma \left(x^\beta\right) \quad . \tag{8.15}$$

Now, from (8.7) and (8.13) we have, for any multi-index σ in $\mathbb{R}^n$ with $\sigma \le \gamma$,

$$\begin{aligned} {\mathcal{C}_i}^{\sigma_i}\left({x_i}^{\beta_i}\right) &= {x_i}^{2(\gamma_i - \sigma_i)} \prod_{j=0}^{\sigma_i - 1} (2\gamma_i - 2j)(2\gamma_i - 2j + 2c_i) \\ &= {x_i}^{2(\gamma_i-\sigma_i)} 2^{2\sigma_i} \frac{\gamma_i!}{(\gamma_i - \sigma_i)!} \frac{(\gamma_i + c_i)!}{(\gamma_i + c_i - \sigma_i)!} \quad , \end{aligned}$$

and therefore

$$\mathcal{C}^\sigma\left(x^\beta\right) = x^{2(\gamma-\sigma)} 2^{2|\sigma|} \frac{\gamma!}{(\gamma-\sigma)!} \frac{(\gamma+c)!}{(\gamma+c-\sigma)!} \quad ,$$

where we set $c = (c_1, \ldots, c_n)$. Substituting this expression into (8.15) yields

$$p_\beta(x,t) = \nu!\,(\gamma+c)! \sum_{\sigma \le \gamma} \binom{\gamma}{\sigma} \frac{2^{2|\sigma|}\,|\sigma|!\,\varepsilon^\sigma x^{2(\gamma-\sigma)} t^{\ell|\sigma|}}{\ell^{\ell|\sigma|}(\nu+|\sigma|)!\,(\gamma+c-\sigma)!} \quad . \tag{8.16}$$

Finally, setting $\ell = 2$ and $\nu = (0,b)$ we arrive at the representation

$$p_\beta(x,t) = b!\,(\gamma+c)! \sum_{\sigma \le \gamma} \binom{\gamma}{\sigma} \frac{\varepsilon^\sigma x^{2(\gamma-\sigma)} t^{2|\sigma|}}{(b+|\sigma|)!\,(\gamma+c-\sigma)!}$$

as our polynomial solution of (8.10) – (8.11), where $\beta = 2\gamma$.

Example 10. *We consider the Cauchy problem*

$$\frac{\partial}{\partial t}u\left(x,t\right)=\sum_{i=1}^{n}\varepsilon_i\left(\frac{\partial^2}{\partial {x_i}^2}+\frac{2c_i+1}{x_i}\frac{\partial}{\partial x_i}\right)u\left(x,t\right)\quad , \tag{8.17}$$

$$u\left(x,0\right)=x^{\beta}\qquad(\beta=2\gamma)\quad . \tag{8.18}$$

Haimo [11] *studied polynomial solutions of this problem in the case* $\varepsilon_i=1$ *for each* i. *We assume only that each* ε_i *is a nonzero complex number. For the operator on the left of (8.17) we have*

$$\mathcal{B}=\frac{\partial}{\partial t}\qquad,\qquad B\left(z\right)=z\qquad,\qquad \nu=0\quad .$$

The analysis of this problem is exactly like that in Example 9, except that in (8.16) we now set $\ell=1$ *and* $\nu=0$ *to obtain the polynomial solution*

$$p_{\beta}\left(x,t\right)=(\gamma+c)!\sum_{\sigma\leq\gamma}\binom{\gamma}{\sigma}\frac{\varepsilon^{\sigma}x^{2(\gamma-\sigma)}\left(4t\right)^{|\sigma|}}{(\gamma+c-\sigma)!}\quad .$$

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